ICOLS99

Innsbruck

LASER SPECTROSCOPY

XIV INTERNATIONAL CONFERENCE

Innsbruck, Austria June 7–11, 1999

Editors

Rainer Blatt

Jürgen Eschner

Dietrich Leibfried

Ferdinand Schmidt-Kaler

Institut für Experimentalphysik
Universität Innsbruck, Austria

World Scientific
Singapore • New Jersey • London • Hong Kong

QUANTUM THEORY OF CLOCKS AND OF GRAVITATIONAL SENSORS USING ATOM INTERFEROMETRY

CHRISTIAN J. BORDÉ

Laboratoire de Physique des Lasers, Université Paris-Nord, Villetaneuse, and Laboratoire de Gravitation et Cosmologie Relativistes, Université Pierre et Marie Curie, Paris, France

Tel +33-608221484, Fax +33-608018459, E-mail: borde@lpl.univ-paris13.fr

Atom interferometers have become remarkable measuring instruments when used as clocks in the microwave or in the optical domain, as inertial sensors or for the determination of atomic masses and of the fine structure constant [1,2,4,14,15,16,17]. The accuracy of these devices is now such that a new theoretical framework is required, which includes:

1 - A fully quantum mechanical treatment of the atomic motion in free space and in the presence of a gravitational field (most cold atom interferometric devices use atoms in "free fall", e.g. in a fountain geometry),

2 - An account of simultaneous actions of gravitational and electromagnetic fields in beam splitters,

3 - A second quantization of the matter fields to take into account their fermionic or bosonic character in order to discuss the role of coherent sources and their noise properties,

4 - A covariant treatment including spin to evaluate general relativistic effects [3,6,11]. One would like also to be able to discuss the propagation of antimatter in interferometers in the presence of gravitation and the properties of coherent antimatter waves (generated by an antiatom laser such as an antihydrogen Bose-Einstein condensate).

The current state-of-the-art of our theoretical description of atom interferometry to meet these requirements is presented. The quantization of the atomic motion in the longitudinal direction leads to a reinterpretation of usual Ramsey fringes, in terms of a velocity change in this direction, as well as in the transverse direction [4,5,6]. To derive accurate formulas for the phase shifts, the propagation of atom fields using the ABCD formalism of Gaussian optics is reviewed [7]. The dispersion surface of the beam splitters is discussed, from which a new look at Rabi oscillations as a pendellösung phenomenon is developed [6,8]. Finally, relativistic phase shifts have been derived, using quantum field theory of atoms with spin, in the presence of weak gravitational fields treated as spin 2 fields in flat space-time [11].

1 Physics of the beam splitter

Quite generally a matter-wave beam splitter uses a scattering process such as:

$$A(M_A, E_A, \vec{p}_A) + B(M_B, E_B, \vec{p}_B) \longrightarrow C(M_C, E_C, \vec{p}_C) + D(M_D, E_D, \vec{p}_D) \quad (1)$$

A is the incoming particle to be scattered into the outgoing particle D through the interaction with quasi-particles B and C of the scattering field. As examples let us mention neutron scattering by phonons: $A \equiv D \equiv n$; two-level atom scattering by photons: $A \equiv$atom in lower state a, $D \equiv$atom in upper state b, in which case C is present only for two-photon processes (Raman or cascade). A special case is obtained if $a \equiv b$, $B \equiv C$ (Bragg scattering of atoms by a standing laser wave). All these particles may have an effective mass M_X or be massless and their energy E_X is related to their momentum $\vec{p}_X$ through a dispersion law. In free space:

$$E_X(\vec{p}) = \sqrt{M_X^2 c^4 + p_X^2 c^2} \quad (2)$$

We can write a Hamiltonian density for the scattering process, which displays its 4-wave mixing character: $g\phi_D^\dagger(x)\phi_C^\dagger(x)\phi_B(x)\phi_A(x) + h.c.$ The corresponding S-matrix expresses energy and momentum conservation through Dirac distributions: $S \propto \delta(E_D + E_C - E_B - E_A)\delta(\vec{p}_D + \vec{p}_C - \vec{p}_B - \vec{p}_A)$. To have a well-defined correspondence between the momenta of the incoming and outgoing particles A and D, we must have a well-defined momentum difference for the quasi-particles $\vec{p}_C - \vec{p}_B$ and, to have a well-defined phase between the corresponding fields, the quasi-particles must be in a coherent state corresponding to a coherent effective field: $V_{eff}(x) \propto \langle\Phi_{BC}|\phi_B(x)\phi_C^+(x)|\Phi_{BC}\rangle + c.c.$. If $E_C = E_B$, this field is time-independent, the scattering is elastic and the Bragg condition is satisfied. More generally, the masses M_A and M_D may be different, in which case the scattering remains elastic and the Bragg condition satisfied if the resonance condition $\hbar\omega_{eff} = E_C - E_B = (M_A - M_D)\, c^2$ is itself satisfied (the direction of the momentum is changed but not its modulus).

To obtain a closed interferometer a number of scattering zones or events are organized so as to form closed paths in space or in space-time with at least two arms, along which the group velocities may differ transversally, as mentioned before, but also longitudinally. For the dispersion law given above, it is clear that a change in momentum will change the group velocity in the longitudinal (forward) direction for massive particles only. This change of the momentum modulus implies a corresponding change in the kinetic energy which can be obtained only with an off-resonant field (either with a time-dependent plane wave or with a monochromatic localized wave). In this case the velocity change will be proportional to the detuning from resonance. The longitudinal momentum borrowed from the diffracting field must be contained in its Fourier expansion and it is always possible to tilt the field angle to adjust continuously the momentum exchange from purely transverse to longitudinal as in the experiments reported in reference 5.

This velocity change along the forward direction is the basis for the so-called mechanical reinterpretation of Ramsey fringes [4,5,6,12,13]. We will now illustrate this point in more detail through a simple first-order theory of Ramsey fringes.

1.1 *First-order theory and reinterpretation of the Ramsey fringes*

Let us consider a beam of two-level atoms with $E_a < E_b$ initially in state a which interacts successively with two field zones respectively centered at x_1 and x_2 and let us calculate the excited state amplitude to first order in each field zone:

$$b^{(1)}(\vec{r},t) = \frac{1}{i\hbar}\int_{-\infty}^{t} dt' \int \frac{d^3p}{(2\pi\hbar)^{3/2}} \int \frac{d^3k}{(2\pi)^{3/2}} V_{ba}(\vec{k},t') e^{i\vec{k}\cdot(\vec{r}-\vec{r_1})}$$
$$e^{i\left[E_b(\vec{p}+\hbar\vec{k})-E_a(\vec{p})\right](t'-t)/\hbar} e^{i[\vec{p}\cdot\vec{r}-E_a(\vec{p})t]/\hbar} \langle a|\langle\vec{p}|\Psi^{(0)}\rangle \quad (3)$$

where the energy is given by the dispersion relation 2 and can be expanded in a Taylor series:

$$E(\vec{p}+\hbar\vec{k}) = E(\vec{p}) + \frac{\hbar\vec{k}\cdot\vec{p}c^2}{E(\vec{p})} + \frac{(\hbar k)^2c^2}{2E(\vec{p})} + \ldots = E(\vec{p}) + \hbar\vec{k}\cdot\vec{v} + \hbar\delta + \ldots \quad (4)$$

If the initial wave packet has a narrow width in momentum around $\vec{p}_0$:

$$b^{(1)}_{\vec{p}_0}(\vec{r},t) = \left[\begin{array}{c} \frac{1}{i\hbar}\int_{-\infty}^{t} dt' \int \frac{d^3k}{(2\pi)^{3/2}} V_{ba}(\vec{k},t') e^{i\vec{k}\cdot(\vec{r}-\vec{r_1})} \\ e^{i\left[E_b(\vec{p}_0+\hbar\vec{k})-E_a(\vec{p}_0)\right](t'-t)/\hbar} \end{array}\right] a^{(0)}_{\vec{p}_0}(\vec{r},t) \quad (5)$$

Let us introduce a monochromatic electromagnetic wave with a Gaussian distribution of k_x [12] and for simplicity let us ignore the dimension y :

$$b^{(1)}_{\vec{p}_0}(\vec{r},t) = i\Omega_{ba} e^{i(kz-\omega t+\varphi)} \frac{w}{2\sqrt{\pi}} \int_{-\infty}^{+\infty} dk_x e^{-\frac{w^2k_x^2}{4}} e^{ik_x(x-x_1)}$$
$$e^{i[\omega-\omega_{ba}-kv_z-k_xv_x-\delta](t-t_1)} \int_{-\infty}^{t} dt' e^{-i[\omega-\omega_{ba}-kv_z-k_xv_x-\delta](t'-t_1)} a^{(0)}_{\vec{p}_0}(\vec{r},t) \quad (6)$$

In the time integral the upper bound t may be extended to infinity if the considered wave packet has left the interaction zone (this can be justified easily and the exact calculation can be found in reference 12). We obtain a δ function expressing energy conservation as expected from the S-matrix: $2\pi\delta(\omega-\omega_{ba}-kv_z-k_xv_x-\delta)$, which annihilates the time precession factor of the pseudo-spin that we have introduced for the illustration in formula 6. If we neglect the k_x-dependence of the recoil shift δ, the effect of this energy conservation is to select a particular spatial Fourier component of wave vector $k_x = (\omega-\omega_{ba}-kv_z-\delta)/v_x$, which clearly appears as a momentum communicated to the atomic wave:

$$b^{(1)}_{\vec{p}_0}(\vec{r},t) = i\frac{\sqrt{\pi}w}{v_x}\Omega_{ba} e^{i(kz-\omega t+\varphi)} e^{-w^2(\omega-\omega_{ba}-kv_z-\delta)^2/4v_x^2}$$
$$e^{i(\omega-\omega_{ba}-kv_z-\delta)(x-x_1)/v_x} a^{(0)}_{\vec{p}_0}(\vec{r},t) \quad (7)$$

hence the Ramsey fringes in the excited state probability:

$$b^{(1)}_{\vec{p}_0}(\vec{r},t) b^{(1)*}_{\vec{p}_0}(\vec{r},t) \propto e^{i(\omega-\omega_{ba}-kv_z-\delta)(x_2-x_1)/v_x} + c.c. \quad (8)$$

In the classical limit, $\hbar \to 0$ for the external motion, and the atom may have both a well-defined position and a well-defined velocity:

$$a^{(0)}_{\vec{p}_0}(\vec{r},t) a^{(0)*}_{\vec{p}_0}(\vec{r},t) \propto \delta\left(\vec{r}-\vec{r_0}-\vec{v}(t-t_0)\right) \quad (9)$$

This classical trajectory for a point-like atom introduces a correspondence between space and time and we retrieve the time-dependence of the off-diagonal matrix element $b^{(1)}_{\vec{p}_0}(\vec{r},t) a^{(0)*}_{\vec{p}_0}(\vec{r},t)$ that we have in the usual non-quantized approach. This correspondence is lost if either the atom wave packet spreading or the recoil effects cannot be neglected anymore ($\hbar^2$ terms in the energy expansion). If we write $\vec{p} = \vec{p}_0 + \hbar\vec{\delta K}$:

$$E(\vec{p}_0 + \hbar\vec{\delta K} + \hbar\vec{k}) \simeq E(\vec{p}_0) + \hbar(\vec{\delta K} + \vec{k})\cdot\vec{v} +$$
$$+ \frac{\hbar^2k^2}{2M} + \frac{\hbar^2\delta K^2}{2M} + \frac{\hbar^2\vec{\delta K}\cdot\vec{k}}{2M} + \ldots \quad (10)$$

The last term will introduce corrections which depend on the spatial derivative of the wave packet. The recoil term $\hbar^2k_x^2/2M$ can be kept in the k_x integration in formula 6 and leads to analytical corrections which I have calculated explicitly. They scale with the ratio of the de Broglie wavelength of the atom to the beam waist and lead to significant asymmetries of the lineshape e.g. in the Doppler-free two-photon lineshape of cold hydrogen. They are also expected to play a role in the accuracy of cold atom clocks in the microwaves.

1.2 *Strong-field theory of the beam splitters*

In order to describe the propagation of the atom waves in a realistic way inside the beam splitters it is necessary to have an approach valid for strong diffracting fields. Such an approach, presented in references [5,6,8,13], is the generalization, to a two-level system, of the dynamical diffraction theory valid in the two-beam approximation for neutrons. It starts with the time-dependent Schroedinger equation in the rotating frame and looks for stationary plane-wave solutions i.e. corresponding to an energy E and to a momentum $\vec{p} = \hbar\vec{K}$. For each wave of wave vector $\vec{K}$ corresponding to the lower state a, there is a coupled wave with the wave vector $\vec{K}+\vec{k}$ corresponding to the excited state b and their amplitudes u are coupled by the equations:

$$\left[\frac{\hbar^2}{2M}(\vec{K}+\vec{k})^2 - E + V_{bb} + \hbar(\omega_{ba}-\omega)\right] u_{b,\vec{K}+\vec{k}} - \hbar\Omega_{ba}u_{a,\vec{K}} = 0$$

$$-\hbar\Omega_{ba}u_{b,\vec{K}+\vec{k}} + \left[\frac{\hbar^2}{2M}\vec{K}^2 - E + V_{aa}\right] u_{a,\vec{K}} = 0 \qquad (11)$$

Apart from the detuning $(\omega_{ba}-\omega)$ these equations are identical to those which describe neutron interferometry. At resonance or in the special case of Bragg diffraction by a standing laser wave ($\omega_{ba} = \omega_{eff} = 0$) the physics is expected to be exactly the same. The internal label a or b is simply added to the external one $\vec{K}$ or $\vec{K}+\vec{k}$. From the compatibility condition of the coupled equations 11 one obtains the dispersion relation for the waves that may propagate in this special crystal made up of light:

$$\left[\frac{\hbar^2}{2M}(\vec{K}+\vec{k})^2 - E + V_{bb} + \hbar(\omega_{ba}-\omega)\right]\left[\frac{\hbar^2}{2M}\vec{K}^2 - E + V_{aa}\right] - \hbar^2\Omega_{ba}^2 = 0 \qquad (12)$$

The corresponding dispersion surface is displayed in reference 8. From the cut by a constant energy plane, one can show that eight waves may propagate in this crystal, four of which are reflected waves and four propagate in the forward direction. They group themselves in two pairs which differ by the momentum $\hbar\vec{k}$ and by their internal state. Within each pair the two solutions (corresponding to the α and β branches of neutron dynamical diffraction theory) differ by a small momentum difference in the forward direction. These waves beat together, producing what is called *pendellösung* oscillations for

neutron waves [20] and Rabi oscillations in atomic physics [18]. As in the case of Ramsey fringes, the quantization of the external motion gives a new picture for the Rabi oscillations, common to a particle without internal structure and to two-level atomic systems.

The dispersion surface 12 can also be used to discuss the group velocity properties within the splitter. The atomic currents exhibit a very peculiar behaviour discussed in references 6 and 8 and lead to the so-called Borrmann effect. The detailed knowledge of these propagation properties (directions of the current flows, effective masses...) within the splitter is essential to be able to calculate properly the phase shifts due to other fields, in actual interferometers.

2 Influence of the gravitation

2.1 *Propagator of the atomic waves in the presence of gravitation: ABCD formalism*

Let us first consider the propagation of atom waves outside the beam splitters. For this purpose one needs the propagator of Schroedinger equation in the presence of the earth gravitational acceleration $\vec{g}(z) = -(g-\gamma z)\,\hat{z}$:

$$i\hbar\frac{\partial|\Psi(t)\rangle}{\partial t} = \left[H_0 + \frac{p_{op}^2}{2M} + Mgz - \frac{M}{2}\gamma z^2\right]|\Psi(t)\rangle \qquad (13)$$

This propagator is derived in great detail in reference 7:

$$K_T(z,z',t,t') = \exp\left[-iH_0(t-t')/\hbar\right]\exp\left[\frac{iM}{\hbar}\dot{\xi}(z-\xi)\right] \exp\left[\frac{i}{\hbar}\int_{t'}^{t}L(t_1)dt_1\right]K(z-\xi,z',t,t') \qquad (14)$$

where K is the propagator $(M/2\pi i\hbar B)^{1/2}\exp\left[(iM/2\hbar B)(Dz^2 - 2zz' + Az'^2)\right]$ which describes the diffraction spreading of the wave packet and the tidal effects within this wave packet and which is given in terms of the Gaussian optics $ABCD$ matrix coefficients :

$$A = D = \cosh\left(\sqrt{\gamma}(t-t_0)\right); B = \frac{1}{\sqrt{\gamma}}\sinh\left(\sqrt{\gamma}(t-t_0)\right); C = \gamma B \qquad (15)$$

Finally: $L(t) = M(\dot{\xi}^2/2 + \gamma\xi^2/2 - g\xi)$ and the gravitational displacement ξ satisfies: $\ddot{\xi} = -g + \gamma\xi$ which yields $\xi = (g/\gamma)\left[1 - \cosh\left(\sqrt{\gamma}(t-t_0)\right)\right]$.

This formalism can be extended to the three space dimensions x, y, z. When this propagator is applied to a complete basis of Hermite-Gauss wave packets in each dimension, the following result is obtained:

$$wave\_packet(z,t) = \exp\left[\frac{iS_{cl}(t,t_0)}{\hbar}\right] wave\_packet_{t_0}(z - z_{cl}, v_{cl}, X, Y) \quad (16)$$

where $S_{cl}(t, t_0)$ is the classical action and where $wave\_packet_{t_0}$ is the wave packet at the initial time t_0 e.g. the lowest order Gaussian mode

$$\frac{1}{\sqrt{X_0}} \exp\left[\frac{iM}{2\hbar}\frac{Y_0}{X_0}(z - z_0)^2\right] \exp\left[iMv_0(z - z_0)/\hbar\right]$$

in which the central position z_0, the initial velocity v_0 and the initial complex width parameters X_0, Y_0 in phase space have to be replaced by their values at time t given by the A, B, C, D transformation law:

$$z_{cl} = Az_0 + Bv_0 + \xi \quad ; \quad X = AX_0 + BY_0$$
$$v_{cl} = Cz_0 + Dv_0 + \dot{\xi} \quad ; \quad Y = CX_0 + DY_0$$

In the limit where $\gamma \to 0$, $A = D \to 1$, $B \to t - t_0$, $C \to 0$ and $\xi \to -(1/2)g(t - t_0)^2$.

When this result is applied, for example, to the atomic clock fountain it has the simple effect to replace $k_x(x_2 - x_1)$ in formula 8 by $k_x\xi$ where k_x is the boost along the excited state parabola. More general formulas are easily derived for more complicated interferometers in the case where $\gamma \neq 0$.

2.2 Influence of gravitation on the beam splitting process

In precision experiments with matter-wave interferometers, one cannot ignore the fact that gravitation also acts on the atom waves within the beam splitter and this was already found to be critical in neutron interferometry. The ideal way to handle this problem is to look for an exact solution of Schroedinger equation with all fields included in the Hamiltonian. Such exact solutions have been found in the case of square temporal laser pulses in the presence of a simple acceleration term accounting for the earth gravity [9,22]. In other cases only approximate solutions such as WKB solutions could be used to derive the phase shifts [10].

3 General relativity and atom interferometry [3,6,11]

It is possible to include all possible effects of inertial fields as well as all the general relativistic effects of gravitation in a consistent and synthetic framework, in which the atomic fields are second-quantized. The starting point is the use of coupled field equations for atomic fields of a given spin in curved space-time: e.g. coupled Klein-Gordon, Dirac or Proca equations. Gravitation is described by the metric tensor $g_{\mu\nu}$ and by tetrads, which enter in these equations. Several strategies can then be adopted: one can perform Foldy-Wouthuysen transformations [19], but conceptual difficulties arise in the case of arbitrary $g_{\mu\nu}$; one can go to the weak-field limit $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$ and use renormalized spinors and finally one can consider $h_{\mu\nu}$ as a spin-two tensor-field in flat space-time [21] and use ordinary relativistic quantum field theory. Using this last approach, we have been able to derive field equations that display all interesting terms, coupling Dirac atomic fields, gravitational and electromagnetic fields and simple expressions of the corresponding relativistic phase shifts in atom interferometers [11].

The evolution equation of the state vector $|\Psi(t)\rangle$ in the interaction picture is

$$i\hbar\frac{d}{dt}|\Psi(t)\rangle = \int d^3x\ \theta^\dagger(x)\mathcal{V}_G(x)\theta(x)\ |\Psi(t)\rangle\,, \quad (17)$$

where the operator $\mathcal{V}_G(x)$, acting on the field operator $\theta(x)$, is given by

$$\begin{aligned}\mathcal{V}_G = &\frac{1}{2}Mc^2\gamma^0h^{00} + \frac{i\hbar c}{8}\partial_k h_{0j}(\gamma^k\gamma^j - \gamma^j\gamma^k)\\ &+\frac{i\hbar c}{4}\gamma^0(2\partial_k h^{0k}\gamma^0 + \partial_k h^{jk}\gamma_j - \partial_k h^{00}\gamma^k)\\ &+\frac{i\hbar c}{2}\gamma^0\left[2h^{0k}\gamma^0 + h^{jk}\gamma_j - h^{00}\gamma^k\right]\partial_k \end{aligned} \quad (18)$$

The free field operator θ is written as: $\theta(x) = \sum_{r=1}^2\int(d^3p)\left[c_r(\overrightarrow{p})\chi^{(+)}_{\overrightarrow{p},r}(x) + d^\dagger_r(\overrightarrow{p})\chi^{(-)}_{\overrightarrow{p},r}(x)\right]$, where $c_r(\overrightarrow{p})$ and $d_r(\overrightarrow{p})$ are the annihilation operators for the particles or antiparticles, respectively, and $\chi^{(\pm)}_{\overrightarrow{p},r}$ are the positive or negative energy solutions of the free Dirac equation:

$$\chi^{(\pm)}_{\overrightarrow{p},r}(x) = \frac{1}{(2\pi\hbar)^{3/2}}\sqrt{\frac{Mc^2}{E(\overrightarrow{p})}}u^{(r)}_{(\pm)}(\overrightarrow{p})e^{\mp i(E(\overrightarrow{p})t - \overrightarrow{p}.\overrightarrow{r})/\hbar} \quad (19)$$

We are interested in the output spinor corresponding to one-particle states: e.g. $\psi(x) = \langle 0|\,\theta(x)\,|\Psi(t)\rangle$ for atoms. The evolution of this spinor is governed by the equation:

$$i\hbar\partial_t\psi = -i\hbar c\gamma^0\gamma^j\partial_j\psi + Mc^2\gamma^0\psi + \mathcal{V}_G(x)\psi \quad (20)$$

to which we may add terms corresponding to diagonal magnetic dipole and off-diagonal electric dipole interactions [3,6]. This equation has been used in reference 6 to discuss all the terms that lead to a phase shift in an interferometer. Here, we will rather use the momentum representation and introduce the Fourier transforms $\tilde{h}_{\mu\nu}(\vec{k})$:

$$i\hbar\partial_t\langle 0|c_B(\vec{p})|\Psi(t)\rangle = \sum_A \int \frac{d^3k}{(2\pi)^{3/2}}\sqrt{\frac{M_Bc^2}{E_B(\vec{p})}}\sqrt{\frac{M_Ac^2}{E_A(\vec{p}-\hbar\vec{k})}}$$

$$u_B^\dagger(\vec{p})V_{BA}(\vec{k},t)u_A(\vec{p}-\hbar\vec{k}) \quad e^{i\left[E_B(\vec{p})-E_A(\vec{p}-\hbar\vec{k})\right]t/\hbar}\langle 0|c_A(\vec{p}-\hbar\vec{k})|\Psi(t)\rangle \quad (21)$$

where the indices B, A may conveniently represent internal states b, a as well as polarization states $r = 1, 2$. To first-order in $\hbar\vec{k}/Mc$:

$$\sqrt{\frac{M_Bc^2}{E_B(\vec{p}+\hbar\vec{k}/2)}}\sqrt{\frac{M_Ac^2}{E_A(\vec{p}-\hbar\vec{k}/2)}}u_B^\dagger(\vec{p}+\hbar\vec{k}/2)V_{BA}(\vec{k},t)u_A(\vec{p}-\hbar\vec{k}/2) =$$

$$\frac{E\tilde{h}_{00}}{2} + \frac{i\hbar\tilde{h}_{00}}{4M(\gamma+1)}(\vec{k}\times\vec{p})\cdot w_B^\dagger\vec{\sigma}w_A$$

$$-c\vec{p}\cdot\tilde{\vec{h}} - \frac{i\hbar c}{4\gamma}w_B^\dagger(\vec{\sigma}_\perp+\gamma\vec{\sigma}_\parallel)w_A\cdot(\vec{k}\times\tilde{\vec{h}}) - \frac{i\hbar c}{2(\gamma+1)M}w_B^\dagger\vec{\sigma}w_A\cdot(\vec{k}\times\vec{p})\frac{\vec{p}\cdot\tilde{\vec{h}}}{E}$$

$$+\frac{c^2}{2E}\vec{p}\cdot\tilde{\overleftrightarrow{h}}\cdot\vec{p} + \frac{i\hbar c^2}{4E\gamma}(\vec{K}\times\tilde{\overleftrightarrow{h}}\cdot\vec{p})\cdot w_B^\dagger\vec{\sigma}w_A \quad (22)$$

where: $M_B = M_A = M$, $\gamma = 1/\sqrt{1-\beta^2}$, $w_{B,A}$ are the Pauli two-component spinors of the Dirac bispinors, $\vec{K} = \vec{k}_\parallel + \gamma\vec{k}_\perp$, the indices $\parallel$ and $\perp$ designate vector parts respectively parallel and perpendicular to $\vec{p}$.

Expression 22 displays again all the terms which may lead to a gravitational phase shift in an interferometer and can be inserted in equation 3 in place of $V_{ba}(\vec{k},t')$ to derive these phase shifts to first order:

-on the first line the terms involving $\tilde{h}_{00}$ lead to the gravitational shift ($h_{00} = -2\,\vec{g}\cdot\vec{r}/c^2$), to shifts involving higher derivatives of the gravitational potential and to the analog of the Thomas precession.

-the next line, which involves $\tilde{\vec{h}} = \{\tilde{h}_{0k}\}$, gives the Sagnac effect in a rotating frame ($\vec{h} = \vec{\Omega}\times\mathbf{r}/c$) including the Lense-Thirring effect through the first term and the spin-rotation coupling and a relativistic correction (analogous to the Thomas term) through the last two coupling terms.

-the third line describes genuine General Relativity effects including gravitational waves and involves the tensor $\tilde{\overleftrightarrow{h}} = \{\tilde{h}_{ij}\}$. The final term is a very small spin-gravitation effect, due to the coupling between spin and space-time curvature.

These expressions are valid for spins 0 and 1/2 and may be conjectured to be valid for arbitrary spin if $\vec{\sigma}/2$ is replaced by the corresponding spin operator $\vec{S}$.

References

1. *Atom Interferometry*, ed. P. Berman, Academic Press (1997).
2. Ch.J. Bordé, in: *Laser Spectroscopy X*, World Scientific (1991) p.239-245.
3. Ch.J. Bordé, A. Karasiewicz and Ph. Tourrenc, Int. J. of Mod. Phys.D **3**, 157 (1994).
4. U. Sterr et al., in [1] and Appl. Phys. B **54**, 341 (1992).
5. Ch.J. Bordé et al., Phys. Lett. A **188**, 187-197 (1994).
6. Ch.J. Bordé, Matter-wave interferometers: a synthetic approach, in [1].
7. Ch. J. Bordé, *Propagation of Laser beams and of atomic systems*, in *Fundamental Systems in Quantum Optics*, J. Dalibard et al. eds, Elsevier (1991) p.287-380
8. Ch.J. Bordé and C. Lämmerzahl, Annalen der Physik 8, 83-110 (1999).
9. C. Lämmerzahl and Ch.J. Bordé, Phys. Lett. A **203**, 59-67 (1995).
10. C. Lämmerzahl and Ch.J. Bordé, *Atom interferometry in gravitational fields: influence of gravitation on the beam splitter,* General Relativity and Gravitation (May 1999).
11. Ch.J. Bordé, J.-C. Houard and A. Karasiewicz, *Relativistic phase shifts for Dirac particles interacting with weak gravitational fields in matter-wave interferometers*, to be published.
12. Ch.J. Bordé et al., Phys. Rev. **14**, 236 (1976).
13. J. Ishikawa, F. Riehle, J. Helmcke and Ch. J. Bordé, Phys. Rev. **A49,** 4794 (1994).
14. F. Riehle, Th. Kisters, A. Witte, J. Helmcke and Ch.J. Bordé, Phys. Rev. Lett. **67**, 177-180 (1991).
15. B.C. Young, M. Kasevich and S. Chu, in [1] and references therein.
16. Ch.J. Bordé, Physics Letters, **A140**, 10 (1989).
17. Ch. J. Bordé et al., Phys. Rev. **A30**, 1836-1848 (1984).
18. Ch. J. Bordé, S. Avrillier, A. van Lerberghe, Ch. Salomon, Ch. Bréant, D. Bassi, and G. Scoles, Appl. Phys. **B28**, 82 (1982).
19. F. W. Hehl and Wei-Tou Ni, Phys Rev. D **42**, 2045-2048 (1990).
20. C. G. Shull, Phys. Rev. Lett. **21**, 1585-1589 (1968).
21. B.M. Barker, S.N. Gupta and R.D. Haracz, Phys. Rev. **149**, 1027 (1966).
22. K.-P. Marzlin and J. Audretsch, , Phys.Rev. A**53**, 1004 (1996).