I. Physics of the Generalized Beam Splitter

The key element of most matter-wave interferometers is a diffractive beam splitter. Ideally, a diffractive beam splitter is a scattering potential for the incident particles, which is spatially periodic with a wave vector \( \mathbf{k}_{\text{eff}} \): \( V(r) \sim \exp(i \mathbf{k}_{\text{eff}} \cdot r + i \Phi) \) + c.c. and hence couples two modes I and II differing only by their momenta, either \( p \) and \( p + \hbar \mathbf{k}_{\text{eff}} \) or \( p \) and \( p - \hbar \mathbf{k}_{\text{eff}} \). This corresponds to the case of the neutron interferometer [1], but also to an atom interferometer using a mechanical grating [2] or off-resonant standing laser waves [3–5]. It is possible to generalize this beam splitter potential to time varying potentials such as \( V(r,t) \sim \exp[-i(\omega_{\text{eff}} t - \mathbf{k}_{\text{eff}} \cdot r + \Phi)] \) + c.c. if the incident particles simultaneously may undergo a change of their internal state \( a \rightarrow b \), states whose respective energies are \( E_a \) and \( E_b \) [6–9] (Fig. 1). If the resonant condition \( \hbar \omega_{\text{eff}} = \hbar \omega_a = E_b - E_a \) is satisfied (we examine later what happens out of resonance), the change of internal state provides only an additional label to the momentum label for channels I and II: I \( \equiv (a,0) \), II \( \equiv (b,\hbar \mathbf{k}_{\text{eff}}) \). We may consider
that the process is still elastic (no extra energy is given to translation) and the Bragg condition results from the conservation of kinetic energy:

\[ \mathbf{k}_{\text{eff}} \cdot (\mathbf{p} + \hbar \mathbf{k}_{\text{eff}}/2) = 0 \]  

from which, the Bragg angle \( \theta_B \) is given by

\[ \sin \theta_B = \frac{\lambda_{\text{dB}}}{2 \lambda_{\text{eff}}} \]  

where \( \lambda_{\text{dB}} \) is the de Broglie wavelength and \( \lambda_{\text{eff}} = 2\pi/k_{\text{eff}} \).

The only condition required to have a large diffraction angle \( \theta_B \) is to have matched wavelengths \( \lambda_{\text{dB}} \approx \lambda_{\text{eff}} \). In neutron interferometers, this matching results from the short interatomic distance in the silicon crystal. In atom interferometers, it can be obtained by an increase of \( \lambda_{\text{dB}} \), which becomes comparable to an optical wavelength for cold atoms or, in the future, by using very short optical wavelengths in the case of atoms and molecules at ordinary (room) temperatures. If this is not the case, the two output channels I and II may not be fully resolved in space, but for a number of applications, this is not an obstacle, since the extra-label of the internal state \( a,b \) may then be used to discriminate between I and II. It should be emphasized that the splitting in space occurs only because of momentum conservation, not because of a change in the internal state, al-
though it may be accompanied by such a change. We thus have a general class of interferometers in which the splitters change, in general, both the external motion and the internal state in a single step. This includes, as a special case, interferometers in which the splitters change only the external motion \((a = b)\). Of course, we have also another class of interferometers, in which the first step is to create a superposition of states (labels), either internal or external (e.g., spin states in Stern–Gerlach interferometers \([10]\)), and the second step is to separate these states in physical space by a diagonal state (label) dependent potential \(V_{aa}(r)\) or \(V_{bb}(r)\). The interferometers of Refs. 6, 8, and 9 do not belong to this second class, contrary to what is stated in Ref. 3.

In addition to the potential's phase, the potential's amplitude may be modulated either spatially or temporally. For example, a rectangular potential profile in space is obtained with silicon crystal splitters in the neutron interferometer and also with suitably shaped cw laser beams. In this case, energy conservation allows only one momentum modulus \(p\) but the Bragg condition is partly relaxed with respect to the direction of \(p\), owing to the width \(\Delta k\) of the spatial Fourier transform of the potential. On the other hand, if the potential envelope is a plane wave rectangularly pulsed in time \([9]\), only one momentum \(\hbar \mathbf{k}_{\text{eff}}\) can be exchanged, but the Bragg condition is partly relaxed because of the Fourier width, \(\Delta \omega\), which allows departure from elasticity. These two cases are not equivalent, and a simple coordinate transformation from the laboratory frame to the atomic frame will not turn one into the other. Rather, one should exchange the roles of the space \(x\) and time \(t\) coordinates, on the one hand, and of the momentum \(p_x\) and energy \(E\), on the other hand. An atom interferometer in space using the first kind of splitter will be turned into an interferometer in space–time with the second kind of splitter.

If, in addition \(\omega_{\text{eff}} \neq \omega_{\text{bar}}\), the Bragg condition no longer is satisfied, but as we shall see, energy conservation is satisfied because the atom acquires a \(k'_{\text{eff}} \neq k_{\text{eff}}\) in the Fourier transform of the splitter potential (within the width \(\Delta k\)) in the case of the spatial pulse and another effective \(\omega'_{\text{eff}} \neq \omega_{\text{eff}}\) (within the Fourier width \(\Delta \omega\)) in the case of the temporal pulse. This will result in extra phase factors correcting the laser carrier phase with a momentum correction in the first case and an energy correction in the second case.

A. THE EQUIVALENT TWO-LEVEL SYSTEM AND THE EFFECTIVE HAMILTONIAN

Most beam splitters used so far in atom interferometry can be described in terms of an effective traveling wave interacting with an effective two-level system, whether single-photon or multiphoton transitions are used,\(^1\) provided that the intermediate states are off-resonance and can be removed adiabatically (Fig. 2).

\(^1\)In Refs. 6 and 7, it is explicitly stated that beam splitters can also use two-photon transitions.
For a two-level system, we shall write the electric dipole Hamiltonian matrix element as

\[ V_{ab} = -\hbar <\Omega_{ba}, U(r,t) \exp[i(\omega t - k \cdot r + \varphi)] > \]  

where \( \Omega_{ba} = \mu_{ab} E_0/(2\hbar) \) is the Rabi frequency and \( U(r,t) \) the field envelope.
For a folded three-level system ($E_a < E_b < E_c$) interacting with two fields, the matrix elements corresponding to the equivalent two-level system are

\[
V_{\alpha\alpha} = \hbar \sum_{j=1,2} |\Omega_{j\alpha\in}|^2 U_j(r,t) U^*_j(r,t) / \Delta \omega_{j\alpha}\in
\]

\[
V_{\beta\beta} = \hbar \sum_{j=1,2} |\Omega_{j\beta\in}|^2 U_j(r,t) U^*_j(r,t) / \Delta \omega_{j\beta}\in
\]

\[
V_{\alpha\beta} = \hbar (\Omega_{1\alpha\in} \Omega_{2\beta\in} / \Delta \omega_{2\beta\in}) U_{\text{eff}}(r,t) \exp[i(\omega_{\text{eff}} t - k_{\text{eff}} \cdot r + \varphi_{\text{eff}})] + (1 \leftrightarrow 2)
\]

\[
V_{\beta\alpha} = \hbar (\Omega_{1\beta\in} \Omega_{2\alpha\in} / \Delta \omega_{1\alpha\in}) U_{\text{eff}}(r,t) \exp[-i(\omega_{\text{eff}} t - k_{\text{eff}} \cdot r + \varphi_{\text{eff}})] + (1 \leftrightarrow 2)
\]

with $\omega_{\text{eff}} = \omega_1 - \omega_2$, $k_{\text{eff}} = k_1 - k_2$, $U_{\text{eff}}(r,t) = U_1(r,t) U_2^*(r,t)$, $\varphi_{\text{eff}} = \varphi_1 - \varphi_2$, and where

\[
\Delta \omega_{j\alpha\beta} = \omega_j - \omega_\alpha\beta - k_j \cdot v - \delta_j
\]

is the detuning from the intermediate state $c$, corrected by the Doppler and recoil shifts. In these equations, field 1, having envelope $U_1(r,t)$, wave vector $k_1$, phase $\varphi_1$, frequency $\omega_1$, and associated Rabi frequency $\Omega_{1\alpha\in}$, drives the $a\rightarrow c$ transition; and field 2, having envelope $U_2(r,t)$, wave vector $k_2$, phase $\varphi_2$, frequency $\omega_2$, and associated Rabi frequency $\Omega_{2\beta\in}$, drives the $b\rightarrow c$ transition. Using the resonance condition, we check that

\[
\Omega_{\text{eff}}^{ba} = -(\Omega_{1\alpha\in} \Omega_{2\beta\in} / \Delta \omega_{2\beta\in}) = -(\Omega_{1\alpha\in} \Omega_{2\beta\in} / \Delta \omega_{1\alpha\in})^*.
\]

These formulas can be generalized easily for higher order interactions. A special case is that of the standing wave of frequency $\omega = \omega_2 = \omega$ for which $\omega_{\text{eff}} = 0$ and $k_{\text{eff}} = 2\pi k$ for the $n$th order Bragg angle. Thus, the diffraction from a standing wave results from the diffraction by an equivalent effective traveling wave with zero frequency. This conclusion applies also to the case of the silicon crystal of the neutron interferometer. Let us point out, however, that there is an interesting new possibility with the standing-wave beam splitter in the atomic case, which is to use polarized light to induce transitions between magnetic sublevels. In this case, the extra labels $a$ and $b$ are replaced by magnetic quantum numbers $M_a$ and $M_a \pm 2n$, which therefore will be different along the two arms of the interferometer without the need for an additional spin flipper as in the neutron interferometer.

Similar formulas hold for the two-photon cascade case ($E_a < E_c < E_b$) [11]:

\[
V_{\alpha\alpha} = \hbar \sum_{j=1,2} |\Omega_{j\alpha\in}|^2 U_j(r,t) U^*_j(r,t) / \Delta \omega_{j\alpha}\in
\]

\[
V_{\beta\beta} = -\hbar \sum_{j=1,2} |\Omega_{j\beta\in}|^2 U_j(r,t) U^*_j(r,t) / \Delta \omega_{j\beta}\in
\]

\[
V_{\alpha\beta} = -\hbar (\Omega_{1\alpha\in} \Omega_{2\beta\in} / \Delta \omega_{2\beta\in}) U_{\text{eff}}(r,t) \exp[i(\omega_{\text{eff}} t - k_{\text{eff}} \cdot r + \varphi_{\text{eff}})] + (1 \leftrightarrow 2)
\]

\[
V_{\beta\alpha} = \hbar (\Omega_{1\beta\in} \Omega_{2\alpha\in} / \Delta \omega_{1\alpha\in}) U_{\text{eff}}(r,t) \exp[-i(\omega_{\text{eff}} t - k_{\text{eff}} \cdot r + \varphi_{\text{eff}})] + (1 \leftrightarrow 2)
\]
with \( \omega_{\text{eff}} = \omega_1 + \omega_2, k_{\text{eff}} = k_1 + k_2, U_{\text{eff}}(r,t) = U_1(r,t)U_2(r,t), \varphi_{\text{eff}} = \varphi_1 + \varphi_2 \). Using again the resonance condition, we check that

\[
\Omega_{ba}^{\text{eff}} = \frac{\Omega_{1ac}^2}{\Omega_{2bc}^2} = \frac{\Omega_{1ac}^2}{\Omega_{2bc}^2} = \frac{\Omega_{1ac}^2}{\Omega_{2bc}^2} = \frac{\Omega_{1ac}^2}{\Omega_{2bc}^2},
\]

In the special case of a standing wave \( \omega_{\text{eff}} = 2\omega \) and \( k_{\text{eff}} = 0 \).

One therefore can vary \( k_{\text{eff}} \) from 0 to \( 2n\pi \) and \( \omega_{\text{eff}} \) from 0 to \( 2\omega \). The theory and the physics to be discussed apply to all these cases. From now on, we shall drop the notation \( \text{eff} \) for \( \omega_{\text{eff}}, k_{\text{eff}}, \varphi_{\text{eff}}, U_{\text{eff}}(r,t), \) and \( \Omega_{ba}^{\text{eff}} \).

**B. TIME-INDEPENDENT TREATMENT OF THE TRAVELING-WAVE BEAM SPLITTER**

A common theoretical approach thus can be given for all cases, including periodic potentials created by a crystalline or by a fabricated structure (this structure should be thick enough for the two-beam approximation to hold). Here, we present only a very simplified version of this theory which is a straightforward generalization of the two-beam dynamical diffraction theory [1].

We start with the time-dependent Schrödinger equation written for the isospinor \( \Psi(r,t) = \begin{bmatrix} \Psi^*_1(r,t) \\ \Psi^*_2(r,t) \end{bmatrix} \) in the Schrödinger representation in the laboratory frame

\[
i\hbar \frac{\partial \Psi(r,t)}{\partial t} = \begin{bmatrix} \hbar^2 \nabla^2 + \mathbf{H}_0 + \mathbf{V}(r,t) \end{bmatrix} \Psi(r,t)
\]

where \( \mathbf{H}_0 \) and \( \mathbf{V} \) are the following Hamiltonian matrices:

\[
\mathbf{H}_0 = \frac{E_a + E_b}{2} \sigma_0 + \frac{\hbar \omega_{ba}}{2} \sigma_3
\]

\[
\mathbf{V}(r,t) = \begin{bmatrix} V_{bb} - \hbar \Omega_{ba} U_k^*(x - x_i) \exp[-i(\omega t - \mathbf{k} \cdot \mathbf{r} + \varphi)] \\ -\hbar \Omega_{ba} U_k(x - x_i) \exp[i(\omega t - \mathbf{k} \cdot \mathbf{r} + \varphi)] V_{aa} \end{bmatrix}
\]

where the rotating-wave approximation has been used and \( \sigma_j \) denotes the Pauli matrices with \( \sigma_0 \) the unit matrix.

To eliminate the time dependence from the Hamiltonian, we perform the following transformation to a frame rotating at frequency \( \omega \):

\[
\psi(r,t) = \exp[i(\omega t + \varphi)] \Psi(r,t).
\]

In the case of a constant field amplitude \( U(x - x_i) = 1 \), we may then look for stationary plane wave solutions:

\[
\psi(r,t) = \begin{bmatrix} \psi_{b,k}(r,t) \\ \psi_{a,0}(r,t) \end{bmatrix} = \begin{bmatrix} \exp[i(K + \mathbf{k}) \cdot \mathbf{r}] & 0 \\ 0 & \exp[iK \cdot \mathbf{r}] \end{bmatrix} \psi(t)
\]
MATTER-WAVE INTERFEROMETERS

\( \hat{\psi}(t) = \begin{bmatrix} u_{b,k}(p) \\ u_{a,0}(p) \end{bmatrix} \exp(-iEt/\hbar) \) \tag{13}

satisfying

\[ i\hbar \frac{\partial \hat{\psi}}{\partial t} = \begin{bmatrix} H_0 - \hbar\omega \sigma_3/2 + \begin{pmatrix} \hbar^2(K+k)^2/2M + V_{bb} & -\hbar\Omega_{ba} \\ -\hbar\Omega_{ba} & \hbar^2K^2/2M + V_{aa} \end{pmatrix} \end{bmatrix} \hat{\psi} = E\hat{\psi} \] \tag{14}

where \( M \) is the atomic mass and

\[ E = \frac{p^2}{2M} + E_a + \frac{\hbar\omega}{2}. \] \tag{15}

Finally, one obtains

\[ \{\hbar^2(K+k)^2/2M - p^2/2M + V_{bb} + \hbar(\omega_{ba} - \omega)u_{b,k}(p) - \hbar\Omega_{ba}u_{a,0}(p) = 0 \]

\[ -\hbar\Omega_{ba}u_{b,k}(p) + \{\hbar^2K^2/2M - p^2/2M + V_{aa}\}u_{a,0}(p) = 0. \] \tag{16}

At resonance or when \( b = a \) and \( \omega = 0 \), these coupled equations are identical to those of the dynamical neutron diffraction \([1]\) within the two-beam approximation. This demonstrates the equivalence of mechanical, crystal, and laser beam splitters at resonance. The indices \( a,b \) are redundant with respect to \( 0,k \).

1. Solutions in the Symmetric Laue Case: Pendellösung and Rabi Oscillations

These equations can be solved for the unknown vector \( K \). One finds the four waves

\[ \psi^{(1,2)}_{b,k}(r,t) = (u_0/(2\sqrt{1 + y^2}) \exp\{i(p + \hbar k) \cdot r + \epsilon_{1,2}(p/cos\gamma)x - Et/\hbar\} \]

\[ \psi^{(1,2)}_{a,0}(r,t) = (u_0/2)(1 + y/\sqrt{1 + y^2}) \exp\{i(p \cdot r + \epsilon_{1,2}(p/cos\gamma)x - Et/\hbar) \} \] \tag{17}

where

\[ \epsilon_{1,2} = \frac{\hbar M\Omega_{ba}}{p^2} (1 \pm \frac{\sqrt{1 + y^2}}{y^2}) = \frac{MV}{p^2}, \quad \cos\gamma = p \cdot \hat{s}/p \] \tag{18}

and where

\[ y = [\hbar(k + 2\hbar k \cdot p)/(2\hbar M) - (\omega - \omega_{ba}) + (V_{bb} - V_{aa})/\hbar \]

\[ 2\Omega_{ba} \] \tag{19}

is the usual parameter of dynamical diffraction theory that determines the degree to which the Bragg condition is violated, corrected by a term involving the detuning in the case of atoms.
The superscripts 1,2 correspond to the \(\alpha\) and \(\beta\) branches of the neutron dynamical diffraction theory \([1, 12, 13]\). Let us point out that the momenta

\[
\hbar \mathbf{K}^{(1,2)} = \mathbf{p}^{(1,2)} = \mathbf{p} + \epsilon_{1,2} \mathbf{p} \cos \gamma
\]  

(20)

are usually very close to \(\mathbf{p}\), since the correction factor is of order \(\hbar/(2a)\), where \(2a\) is the thickness of the splitter that is to be compared with the de Broglie wavelength. All calculations are performed to the first order in \(\epsilon\). Exact calculations are possible but they require the consideration of reflected waves, which we have ignored here for the sake of simplicity.

Combining Eqs. (11) – (20), one obtains

\[
\Psi_{\text{II}}(\mathbf{r}, t) = i a_0/\sqrt{1 + \gamma^2} \sin [\Omega_{ba} \sqrt{1 + \gamma^2} x/v_x] \exp \left[-i(\omega t - \mathbf{k} \cdot \mathbf{r} + \varphi)\right] \\
\times \exp \left[i(\mathbf{p} \cdot \mathbf{r} - (E_a + p^2/(2M))t)/\hbar \right] \exp \left[-i(V_{ad} \hbar + \Omega_{ba} y)x/v_x \right] \\
\times \exp \left[i[\mathbf{p} \cdot \mathbf{r} - (E_a + p^2/(2M))t]/\hbar \right] \exp \left[-i(V_{ad} \hbar + \Omega_{ba} y)x/v_x \right] \\
\times \exp \left[i(\mathbf{p} \cdot \mathbf{r} - (E_a + p^2/(2M))t)/\hbar \right] \exp \left[-i(V_{ad} \hbar + \Omega_{ba} y)x/v_x \right] \\
\times \exp \left[i(\mathbf{p} \cdot \mathbf{r} - (E_a + p^2/(2M))t)/\hbar \right] \exp \left[-i(V_{ad} \hbar + \Omega_{ba} y)x/v_x \right] \\
\times \exp \left[i(\mathbf{p} \cdot \mathbf{r} - (E_a + p^2/(2M))t)/\hbar \right] \exp \left[-i(V_{ad} \hbar + \Omega_{ba} y)x/v_x \right] \\
\times \exp \left[i(\mathbf{p} \cdot \mathbf{r} - (E_a + p^2/(2M))t)/\hbar \right] \exp \left[-i(V_{ad} \hbar + \Omega_{ba} y)x/v_x \right]
\] 

(21)

with \(v_x = p_x/M\).

For the same problem with a square pulse in the time domain (14), the solution for \(\Psi_{\text{II}}(\mathbf{r}, t)\) is

\[
\Psi_{\text{II}}(\mathbf{r}, t) = a_0/\sqrt{1 + \gamma^2} \sin [\Omega_{ba} \sqrt{1 + \gamma^2} x/v_x] \exp \left[-i(\omega t - \mathbf{k} \cdot \mathbf{r} + \varphi)\right] \\
\times \exp \left[i(\mathbf{p} \cdot \mathbf{r} - (E_a + p^2/(2M))t)/\hbar \right] \exp \left[-i(V_{ad} \hbar + \Omega_{ba} y)x/v_x \right] \\
\times \exp \left[i(\mathbf{p} \cdot \mathbf{r} - (E_a + p^2/(2M))t)/\hbar \right] \exp \left[-i(V_{ad} \hbar + \Omega_{ba} y)x/v_x \right] \\
\times \exp \left[i(\mathbf{p} \cdot \mathbf{r} - (E_a + p^2/(2M))t)/\hbar \right] \exp \left[-i(V_{ad} \hbar + \Omega_{ba} y)x/v_x \right] \\
\times \exp \left[i(\mathbf{p} \cdot \mathbf{r} - (E_a + p^2/(2M))t)/\hbar \right] \exp \left[-i(V_{ad} \hbar + \Omega_{ba} y)x/v_x \right] \\
\times \exp \left[i(\mathbf{p} \cdot \mathbf{r} - (E_a + p^2/(2M))t)/\hbar \right] \exp \left[-i(V_{ad} \hbar + \Omega_{ba} y)x/v_x \right]
\] 

(22)

We see that the two expressions correspond to each other with an exchange of the roles of time and space coordinate \(x\) and energy and momentum \(p_x\).

The energy oscillations between the primary and reflected beams are known as the pendellösung oscillations in the case of neutrons \([15]\) and as Rabi oscillations in the case of atoms. In the case of atoms, the extra label of the internal energy can be used to monitor these oscillations even if the Bragg angle is too small to resolve the beams by their scattering angles (Fig. 3) \([16, 17]\).

2. Neutron or Atomic Currents: A Wave Packet Approach

From the plane wave solutions Eq. (21), one can build wave packets by integration over a momentum distribution \(u(\mathbf{p} - \mathbf{p}_0)\) and look for the trajectories of these wave packets. Each of these has the following structure:

\[
F(\mathbf{r}, t) = \int d^3p \exp \left[-ip^2t/(2M\hbar)\right] \exp \left[i\mathbf{p} \cdot \mathbf{r}/\hbar + i\varphi(\mathbf{p})x/\hbar\right] u(\mathbf{p} - \mathbf{p}_0)
\] 

(23)

where \(\varphi(\mathbf{p})\) stands for \(\epsilon_{1,2} p/\cos \gamma = \epsilon_{1,2} p^2/p_x\).
Fig. 3. Rabi oscillations for a two-level atomic system: A molecular (SF₆) beam crosses a perpendicular cw traveling (CO₂) laser beam at resonance [16, 17]. The excited state population is monitored after the interaction zone and plotted versus the laser electric field amplitude. The two output channels, I and II, are not resolved angularly in this case, because the molecules travel too fast to have a de Broglie wavelength comparable to the laser wavelength, but the extra label provided by the internal (vibrational) state is used to discriminate between channels I and II. This experiment with molecules is equivalent to Shull's pendelösung experiment with neutrons [15].

We can introduce the true momentum $\mathbf{p}_I = \mathbf{p} + \varphi(p)\mathbf{\hat{x}}$ with a change of variable:

$$
\mathbf{p} = \mathbf{p}_I = \varphi_1(\mathbf{p}_I)\mathbf{\hat{x}} \\
\varphi_1(\mathbf{p}_I) = \varphi(\mathbf{p})
$$

(24)

which gives to $F$ the canonical structure

$$
F(r,t) = \int d^3p_1 \exp \left[ -i(\mathbf{p}_I - \varphi_1(\mathbf{p}_I)\mathbf{\hat{x}})^2t/(2M\hbar) \right] \\
\cdot \exp \left[ i\mathbf{p}_I \cdot \mathbf{r}/\hbar \right] \exp \left[ i\mathbf{\hat{x}}\varphi_1(\mathbf{p}_I)t/(M\hbar) \right] \\
\cdot \exp \left[ i(\mathbf{p}_I - \mathbf{p}_0) \cdot \nabla_{p_0}[v_x\varphi_1(\mathbf{p}_I)]/\hbar \right] \exp \left[ i\varphi(\mathbf{p}_0)t/\hbar \right] \\
\cdot \exp \left[ i\mathbf{p}_0 \cdot (\mathbf{r} - \mathbf{p}_0t/M) \right] f(\mathbf{r} - \mathbf{p}_0t/M + [\nabla_{p_0}(v_x\varphi_1(\mathbf{p}_I))]_{p_0}t) \\
\cdot \exp \left[ i(\mathbf{p}_0 - \mathbf{p}_0') \cdot \mathbf{r}/\hbar \right] \exp \left[ -i(\mathbf{p}_0^2 - \mathbf{p}_0'^2t)/(2M\hbar) \right]
$$

(25)

from which we extract the central component at $\mathbf{p}_0'$ with a Taylor expansion around $\mathbf{p}_0'$:

$$
F(r,t) = \exp \left[ -i\mathbf{p}_0'^2t/(2M\hbar) \right] \exp \left[ i\mathbf{p}_0' \cdot \mathbf{r}/\hbar \right] \exp \left[ i\mathbf{\hat{x}}\varphi_1(\mathbf{p}_0')t/(M\hbar) \right] \\
\cdot \int d^3(p - p_0') \exp \left[ i(p - p_0') \cdot \mathbf{r}/\hbar \right] \exp \left[ -i(p_0^2 - p_0'^2t)/(2M\hbar) \right]
$$

(26)
where \( \mathbf{p}'_0 = \mathbf{p}_0 + \varphi_1(\mathbf{p}_0) \hat{\mathbf{x}} = \mathbf{p}_0 + \varphi(\mathbf{p}_0) \hat{\mathbf{x}} \) and where

\[
f(\mathbf{r}) = \int d^3p \exp \left[ i(\mathbf{p} - \mathbf{p}_0) \cdot \mathbf{r}/\hbar \right] u(\mathbf{p} - \mathbf{p}_0)
\]

is the envelope of the wave packet. We check from this expression that the envelope travels at the group velocity and the overall phase factor is given by the action integral and by the dephasing introduced by the splitter taken at the value of the central momentum. The group velocity can also be calculated directly from the dispersion relation:

\[
E(\mathbf{p}_1) = [\mathbf{p}_1 - \varphi_1(\mathbf{p}_1) \hat{\mathbf{x}}]^2/(2M)
\]

and \( \mathbf{v}_g = \nabla_{\mathbf{p}_1} \cdot E(\mathbf{p}_1) \).

The corresponding wave packet structure obtained from Eq. (22), in the time domain, can be written as

\[
F(\mathbf{r}, t) = \int d^3p \exp \{-i[p^2 - 2px\varphi(\mathbf{p})]/(2M\hbar)\} \exp \{i\mathbf{p} \cdot \mathbf{r}/\hbar\} u(\mathbf{p} - \mathbf{p}_0)
\]

The group velocity is the same as in the previous case, since to the first order

\[
\{\nabla_{\mathbf{p}} [v_x \varphi(\mathbf{p})] \}_{\mathbf{p}_0} \sim \{\nabla_{\mathbf{p}} [v_x \varphi(\mathbf{p})] \}_{\mathbf{p}_0}.
\]

The main difference is that the momentum distribution in this case corresponds to an energy distribution in the previous case. If the same manipulation is performed as in Eq. (26), one finds that the spatial phase factor \( \exp \{i\varphi(\mathbf{p}_0)x/\hbar\} \) is replaced by the temporal phase factor \( \exp \{iv_x(\mathbf{p}_0)t/\hbar\} \).

One finds the following expressions for the group velocities:

\[
\mathbf{v}_g^{(1,2)} = \frac{\mathbf{p}^{(1,2)}}{M} + \frac{\hbar \mathbf{k}}{2M} \left( 1 \mp \frac{y}{\sqrt{1 + y^2}} \right)
\]

where all functions are evaluated for the center of the wave packet \( \mathbf{p}_0 \).

This means that, inside the splitter, the incident wave splits into two groups of two waves (the \( \alpha \) and \( \beta \) branches) whose trajectories are symmetrically located on both sides of the normal to the input face of the splitter, making an angle with this normal \( \Omega_{1,2} \) given by

\[
\tan \Omega_{1,2} = \pm \frac{y}{\sqrt{1 + y^2}}.
\]

For incidence at the Bragg angle and at resonance in the case of atoms, this

\[3\]From Eq. 31, one can show easily that the tangential components of the velocities satisfy \( \mathbf{k} \cdot \mathbf{v}_g = \pm (\mathbf{k} \cdot \mathbf{p}_0/M)y/\sqrt{1 + y^2} \), while the normal components are nearly the same.
angle is equal to $0$ and all waves propagate perpendicularly to the input surface. Any small deviation from the Bragg incidence angle is greatly amplified [18], and for atoms, a detuning will have the same effect.

One consequence of these separate trajectories is a correction to the phase shift that one could calculate from a naive theory ignoring them. This has been the case in the calculations of the gravitational phase shift in the neutron Collela, Overhauser, Werner (COW) [45] experiment, where important corrections have been found [13]. Also, the existence of four waves within the beam splitter gives rise to four waves leaving the splitter, and this multiplies the number of interferometers to be considered in phase-shift calculations [19].

3. Inelastic versus Elastic Scattering

An elegant way to introduce the boundary conditions between field zones and field-free zones is to extend the previous analysis to varying field envelopes $U$. In this case $K_x$ is replaced by $-i\partial/\partial x$ in Eqs. (16), which become

$$
\begin{align*}
\{(\hbar^2/2M)\frac{d^2}{dx^2} + p_x^2/2M + \\
\hat{h}
\begin{bmatrix}
\omega - \omega_{ba} - kv_z - \delta - V_{bb}(x - x_i)/\hbar & \Omega_{ba} U^*(x - x_i)/\hbar \\
\Omega_{ab} U(x - x_i) & -V_{aa}(x - x_i)/\hbar
\end{bmatrix}
\end{align*}
\hat{\psi}(x,t) = 0
\tag{33}
$$

Fig. 4. Neutron or atomic currents in the beam splitter: Under Bragg incidence and at resonance the particles travel perpendicularly to the input surface of the splitter. Otherwise, the beam splits along two trajectories within the so-called Borrmann fan, with an angle $\Omega$, which is greatly amplified in comparison with the deviation from the Bragg angle $\theta_B$. At the output surface, these two waves split again in two free-space solutions in channels I and II.
whose solution is

\[ \psi(x,t) = \exp \left[ \frac{i}{\hbar} \left[ p_\alpha(x - x_0) - E(t - t_0) \right] \right] T \exp \left[ i \int_{x_0}^{x} \frac{dx'}{v_x} \right] \]

\[ \left[ \begin{array}{c}
\omega - \omega_{ba} - kv_z - \delta - V_{ba}(x' - x_1) / \hbar \\
\Omega_{ba} U^\dagger(x' - x_1)
\end{array} \right] \psi(x_0,t_0) \] (34)

where

\[ \delta = \frac{\hbar k^2}{2M} \] (35)

is the recoil shift and \( T \) is an ordering operator. Finally,

\[ \Psi_{II}(r,t) = \exp \left[ \frac{i}{\hbar} (p + \hbar k) \cdot (r - r_2) \right] \]

\[ \exp \left[ i(\omega - \omega_{ba} - kv_z - \delta) \frac{x - x_2}{v_x} \right] \Psi_{II}(r_2,t) \] (36)

where \( \Psi_{II}(r_2,t) \) corresponds to the solution given by Eq. (21) in the case of a constant field which cancels at \( r = r_2 \). From this expression, we see that an additional momentum \( \hbar \delta k_x = (\omega - \omega_{ba} - kv_z - \delta) / v_x \) is transferred to the atomic wave along the \( x \) axis. The origin of this extra momentum lies in the inelastic character of the scattering when the effective laser frequency is out of resonance (Fig. 5) [20, 21]. In this case, the additional energy transferred to the atoms is converted into kinetic energy, hence a boost of the atoms along the diffracted path. This is a new feature of the generalized beam splitter that could not be obtained with silicon crystals or mechanical gratings.

The additional momentum \( \hbar \delta k_x \) is easily found by invoking conservation of energy for a two-level system in the laboratory frame (see formulas (61) and (64) in Ref. [11]). Using the relationship \( E = p \cdot v - L \), where \( L \) is the Lagrangian, one finds

\[ (p_b - p_a) \cdot v / \hbar = \omega - \omega_{ba} \sqrt{1 - \beta^2} - \delta \] (37)

where \( \beta = v / c \), and \( v = p_a \sqrt{1 - \beta^2} / M \) [11].

Since, in each zone, the interaction may involve many photon exchanges, the overall momentum \( \hbar k' \) exchanged in Eq. (37) results from these multiple successive one-photon exchanges. This overall exchanged momentum is written as \( (p_b - p_a) = \hbar (k + \delta k) \). Neglecting \( \delta k_x v_x \) in Eq. (37), we infer that the correction \( \delta k_x \) is given by

\[ \delta k_x v_x = \Delta - \delta - k \cdot v \] (38)

where \( \Delta = \omega - \omega_{ba} \sqrt{1 - \beta^2} \) is the detuning corrected by the transverse Doppler effect.
Fig. 5. Inelastic versus elastic scattering or how to communicate momentum along one of the interferometer arms: At resonance, energy conservation requires elasticity in the splitting process and the Bragg condition has to be satisfied. The exchanged momentum $\hbar k'$ can have any value within the spatial Fourier content $\hbar k \pm \hbar \Delta k$ of the splitter profile. On the $x$ axis perpendicular to the splitter, this gives a narrow distribution of momenta, but for each value of the momentum of the incoming particle, the modulus of the momentum is the same on the two output channels. Out of resonance, the additional amount of exchanged energy (positive or negative) is transferred to kinetic energy and the modulus of the momentum is changed in channel II. To the first order, the increase of momentum along the $x$ axis is proportional to the detuning $\Delta$. The exchanged momentum $\hbar k'$ still has to lie within the spatial Fourier content $\hbar k \pm \hbar \Delta k$ of the splitter profile and this limits the possible detuning. It is however possible to tilt $k$ to increase this range and to transfer the recoil continuously from the transverse $\varepsilon$ direction to the longitudinal $x$ direction.
To the second order, there is also a small momentum correction $\hbar \delta k_z$ along the $z$ axis, which is obtained as a solution of the following equation:

$$2\delta k_z = -[(\delta k_x)^2 + (\delta k_y)^2]/k$$

(39)

derived using $k' = k$.

If one solves the same problem in the case of a pulse in the time domain, the corresponding solution would be

$$\Psi_{II}(r,t) = \exp \left[ -\frac{i}{\hbar} \left( E_a + p^2/(2M) + \hbar \omega (t - t_2) \right) \right]$$

$$\exp \left( i(\omega - \omega_{ba} - kv_z - \delta) (t - t_2) \right) \Psi_{II}(r,t_2)$$

(40)

where $\Psi_{II}(r,t_2)$ is given by (22) in the case of a square pulse.

The exchanged momentum $(p_b - p_a = \hbar k)$ is fixed and the extra energy coming from the inelastic character appears as a change in energy along the diffracted path. From the energy–momentum balance (37), one finds, this time, that the effective energy exchanged is

$$\hbar \omega' = \hbar \omega_{ba} \sqrt{1 - \beta^2} + \hbar \delta + (p_b - p_a) \cdot v$$

(41)

hence the correction factor to the laser carrier frequency $\omega$. This is again an illustration of the correspondence between the two types of problems.

C. SCATTERING MATRIX IN THE TIME-DEPENDENT APPROACH AND PROPAGATORS BETWEEN FIELD ZONES

We can unify the cases of spatial and temporal pulses in a more general approach based on a time-dependent description of the evolution of wave packets. The basis of this treatment is the use of the interaction representation with respect to $H_0$ and those parts of the Hamiltonian connected only with the external motion

$$|\tilde{\Psi}(t)\rangle = U_0(t,t_1)|\Psi(t)\rangle$$

(42)

where

$$U_0(t,t_1) = U_{\text{ext}}(t,t_1) \exp \left[ -iH_0(t - t_1)/\hbar \right]$$

(43)

is the free evolution operator in the absence of $V$ in which $U_{\text{ext}}(t,t_1)$ describes the external motion.

The transformed ket $|\tilde{\Psi}(t)\rangle$ satisfies

$$i\hbar \partial_t |\tilde{\Psi}(t)\rangle = \tilde{V}(r_{op},p_{op},t)|\tilde{\Psi}(t)\rangle$$

(44)
with
\[ \tilde{V}(\mathbf{r}_\text{op},\mathbf{p}_\text{op},t) = U^{-1}_0(t,t_1)\mathbf{V}(\mathbf{r}_\text{op},t)U_0(t,t_1) \]
\[ = \exp \left[ iH_0(t - t_1)/\hbar \right] \mathbf{V}[\mathbf{r}_\text{op}(t,t_1),t] \exp \left[ -iH_0(t - t_1)/\hbar \right] \]
\[ = \tilde{V}[\mathbf{r}_\text{op}(t,t_1),t] \quad (45) \]
and
\[ \mathbf{r}_\text{op}(t,t_1) = U^{-1}_E(t,t_1)\mathbf{r}_\text{op}U_E(t,t_1). \quad (46) \]

For expressions for the evolution operators $U_0(t,t')$ in the presence of inertial fields see Refs. 7 and 22.

In free space, $U_E(t,t_1)$ reduces to $\exp \left[ -i(p^2/2M)(t - t_1)/\hbar \right]$, in which case,
\[ \mathbf{r}_\text{op}(t,t_1) = \mathbf{r}_\text{op} + \mathbf{p}_\text{op}(t - t_1)/M. \quad (47) \]

The formal solution of Eq. (44) is ($\mathcal{T}$ is a time-ordering operator)
\[ |\tilde{\Psi}(t)\rangle = \mathcal{T}\left( \mathcal{T}^\dagger V(t)\mathcal{O}(t) \right) |\tilde{\Psi}(t_0)\rangle \]
\[ = \mathcal{T}\exp \left\{ -i \int_{t_0}^t dt' \tilde{V}[\mathbf{r}_\text{op}(t',t_1),t] \right\} |\tilde{\Psi}(t_0)\rangle. \quad (48) \]

1. The $\tilde{S}$ Matrix in the Two-Beam Approximation

Within the two-beam approximation, we shall consider the following isospinor, in the position representation
\[ \langle \mathbf{r}|\tilde{\Psi}(t)\rangle = \tilde{\Psi}(\mathbf{r},t) = \begin{pmatrix} \tilde{\Psi}_{\text{bb}}(\mathbf{r},t) \\ \tilde{\Psi}_{\text{ab}}(\mathbf{r},t) \end{pmatrix}. \quad (49) \]

A great simplification occurs if we calculate the matrix elements of $U(t,t_0)$ with the assumption of very narrow (quasi-plane) wave packets in momentum space, with a correction for the recoil
\[ \langle \mathbf{r}|\tilde{U}(t,t_0)|\mathbf{r}'\rangle = \mathcal{T}\exp \left[ -i \int_{t_0}^t dt' \mathcal{V}(t')/\hbar \right] \delta(\mathbf{r} - \mathbf{r}') \]
\[ = \tilde{S}(\mathbf{r},t_1)\delta(\mathbf{r} - \mathbf{r}') \quad (50) \]
where Hamiltonian matrix $\mathcal{V}(t)$ is given by
\[ \mathcal{V}(t) = \begin{bmatrix} V_{\text{bb}}[\mathbf{r} + (\mathbf{p} + \hbar\mathbf{k}/2)(t - t_1)/M,t] & V_{\text{ba}}^+[\mathbf{r} + \mathbf{p}(t - t_1)/M,t] \\ V_{\text{ab}}[\mathbf{r} + (\mathbf{p} + \hbar\mathbf{k}/2)(t - t_1)/M,t] & \times \exp[i\omega_{\text{ba}}(t - t_1)] \\ \times \exp[-i\omega_{\text{ba}}(t - t_1)] & V_{\text{aa}}[\mathbf{r} + \mathbf{p}(t - t_1)/M,t] \end{bmatrix}. \quad (51) \]
($V_{\text{ba}}^+$ and $V_{\text{ab}}^-$ are the $\exp(i\mathbf{k} \cdot \mathbf{r})$ and $\exp(-i\mathbf{k} \cdot \mathbf{r})$ parts of $V_{\text{ba}}$ and $V_{\text{ab}}$, respectively).
A simple way to derive this result is to start with Eq. (44) in the position representation:

$$i\hbar \partial_t \tilde{\Psi}_{b,k}(r,t) = \int d^3p' \langle r | \tilde{V}_{ba}[r_{op} + p_{op}(t - t_f)/M,t] | p' \rangle \langle p' |\langle a | \tilde{\Psi}(t) \rangle$$

$$+ \int d^3p' \langle r | \tilde{V}_{hh}[r_{op} + p_{op}(t - t_f)/M,t] | p' \rangle \langle p' |\langle b | \tilde{\Psi}(t) \rangle \rangle$$

$$= 1/(2\pi\hbar)^{3/2} \int d^3p' \exp [i \mathbf{p}' \cdot \mathbf{r}/\hbar] \int d^3k' \tilde{V}^+_b(k',t) \exp \{ik' \cdot [\mathbf{r} + \mathbf{p}'(t - t_f)/M] \} \exp [i \hbar k'^2(t - t_f)/2M] \langle p' |\langle a | \tilde{\Psi}(t) \rangle \rangle$$

$$+ 1/(2\pi\hbar)^{3/2} \int d^3p' \exp [i \mathbf{p}' \cdot \mathbf{r}/\hbar] \int d^3k' \tilde{V}_{hh}(k',t)$$

$$\exp [i \mathbf{k}' \cdot [\mathbf{r} + \mathbf{p}'(t - t_f)/M] \langle p' \rangle \langle b | \tilde{\Psi}(t) \rangle \rangle. \quad (52)$$

where we have introduced the spatial Fourier transforms

$$\tilde{V}(r,t) = 1/(2\pi)^{3/2} \int d^3k' \tilde{V}(k',t) \exp [i \mathbf{k'} \cdot \mathbf{r}] \quad (53)$$

and where we have neglected the recoil for the diagonal terms.

Finally, with the assumption that \( \tilde{\Psi}_{a,0}(r,t) \) and \( \tilde{\Psi}_{b,k}(r,t) \) represent narrow wave packets centered around \( \mathbf{p} \) and \( \mathbf{p} + \hbar \mathbf{k} \), one finds

$$i\hbar \partial_t \tilde{\Psi}_{b,k}(r,t) = \tilde{V}_{ba}[\mathbf{r} + (\mathbf{p} + \hbar \mathbf{k}/2)(t - t_f)/M,t] \tilde{\Psi}_{a,0}(r,t)$$

$$+ \tilde{V}_{hh}(\mathbf{r} + \mathbf{p}(t - t_f)/M,t) \tilde{\Psi}_{b,k}(r,t) \quad (54)$$

and a similar equation for \( \tilde{\Psi}_{a,0}(r,t) \).

As an example, in the case of the electric dipole interaction, \( \mathcal{V}(t) \) is given by [23]

$$\mathcal{V}(t) = \begin{bmatrix}
V_{ba}[x - x_c + v_x(t - t_f),t - t_f] & -\hbar \Omega_{ba} U^*[x - x_c + v_x(t - t_f),t - t_f] \\
-\hbar \Omega_{ba} U[x - x_c + v_x(t - t_f),t - t_f] & \times \exp \{ -i[\omega t - k(z + v_z(t - t_f)) + \phi] \} \\
\times \exp \{ i(\omega t - k[(z + v_z(t - t_f)) + \phi]) \} & \times \exp \{ i(\omega_{ba} + \delta)(t - t_f) \} \\
\times \exp \{ i(\omega_{ba} + \delta)(t - t_f) \} & V_{aa}[x - x_c + v_x(t - t_f),t - t_f] \\
\end{bmatrix} \quad (55)$$

Expressions for the matrix \( \tilde{S}(r,t_i) \) can be found for spatial pulses in Refs. 20 and 23. From Eq. (50) and from the previous expression, one finds

$$\tilde{S}(r,t_i) = \exp \{ -i(\omega t - \mathbf{k} \cdot \mathbf{r} + \phi)\sigma_3/2 \} \exp \left[ i(\omega - \omega_{ba} - kv_z - \delta) \frac{x - x_c}{v_x} \sigma_3/2 \right] M \quad (56)$$
where the matrix $M$ is given by

$$M = T \exp \left[ \int_{-\infty}^{+\infty} dt \begin{bmatrix} -V_{bb}(v,t)/\hbar & \Omega_{bd} U^*(v,t) \times \exp[-i(\Delta - kv_z - \delta)t] \\ \Omega_{bd} U(v,t) \times \exp[i(\Delta - kv_z - \delta)t] & -V_{ad}(v,t)/\hbar \end{bmatrix} \right]$$

(57)

which, in the case of fields having rectangular spatial profiles of length $2a$,

$$U(x - x_c) = Y[(x - x_c)/a + 1] - Y[(x - x_c)/a - 1]$$
$$V_{ij}(x - x_c) = [Y[(x - x_c)/a + 1] - Y[(x - x_c)/a - 1]]_{ij}$$

(58)

where $Y$ is a step function, can be expressed as

$$M(A,C') = \exp \left[ -i(V_{bb} + V_{aa})/\hbar (a/v_x) \right] \cos A - i(C'/A) \sin A \exp iC' \times \begin{bmatrix} 2i(A/\Lambda) \sin \Lambda & 2i(A/\Lambda) \sin \Lambda \\ 2i(A/\Lambda) \sin \Lambda & (\cos \Lambda + i(C'/\Lambda) \sin \Lambda) \exp(-iC') \end{bmatrix}$$

(59)

where $A = \Omega_{bd} a/\hbar v_x$, $C' = C' + (V_{bb} - V_{aa})/\hbar (a/v_x)$, $C' = -(\Delta - kv_z - \delta)a/v_x = C + \delta a/v_x$, $C = -(\Delta - kv_z)a/v_x$, and $\Lambda = \sqrt{4A^2 + C'^2}$.

Similar expressions valid for temporal pulses can be derived from Eq. (50), which are consistent with Eq. (40); that is,

$$\tilde{S}(r,t) = \exp \left[ -i(\omega t - k \cdot r + \varphi)\sigma_y/2 \right]$$
$$\exp \left[ i(\omega - \omega_{ba} - kv_z - \delta)(t - t_c)\sigma_y/2 \right] M(A,C)$$
$$\exp \left[ i(\omega t - k \cdot r + \varphi)\sigma_y/2 \right] M(A,C)$$

(60)

This approach can be used also to connect the formulas obtained for temporal [14, 24] and spatial splitters in the presence of a gravitational field. In this case, the free evolution operator $U_{g}(t,t_1)$ given in Refs. 7 and 22 is used to calculate

$$r_{op}(t,t_1) = r_{op} + p_{op}(t - t_1)/M + \xi(t,t_1)$$

(61)

which shifts the spatial coordinates in $V(t)$ by

$$\xi(t,t_1) = g(t - t_1)^2/2$$

(62)

in the case of a constant field $g$. The $\tilde{S}(r,t)$ matrix is then calculated with either $U(t - t_c)$ or $U(x - x_c)$. In this way, the problem that we mentioned previously, of having to calculate the gravitational phase shift along the trajectories in the beam splitter can be avoided.
2. Free Propagators and Calculation of Phase Shifts

The general idea is then to write the evolution operator \( U(t, t_0) \) as a product of free evolution operators and of \( \tilde{S} \) matrices, which give the evolution in the beam splitters:

\[
U(t, t_0) = U_0(t, t_1)U_0^{-1}(t, t_1)U(t, t_0)U_0^{-1}(t_1, t_0)U_0(t, t_0) = U_0(t, t_1)\tilde{U}(t, t_0)U_0(t_1, t_0). \tag{63}
\]

We have then

\[
\langle r | \Psi(t) \rangle = \int d^3r_1 d^3r_0 G_0(r, r_1, t_1, t_1)\tilde{S}(r_1, t_1)G_0(r_1, r_0, t_1, t_0)\langle r_0 | \Psi(t_0) \rangle \tag{64}
\]

where \( G_0(r, r', t, t') = \langle r | U_0(t, t') | r' \rangle \) is the propagator outside the field zones. The expressions for the propagators in the presence of inertial fields can be found in Refs. 7 and 22. The matrix elements of the free-space propagator are given by

\[
G_{a\beta}(r, r', t, t') = \langle r, \alpha | \exp[-i(H_0 + p_{op}^2/2M)(t - t')/\hbar] | r', \alpha \rangle. \tag{65}
\]

For a wave packet centered about \( p_1 \), these matrix elements can be approximated by [22]

\[
G_{a\beta}(r, r', t - t') = \exp[-i(E_a - p_{1}^2/2M)(t - t')/\hbar]\delta[r - r' - p_1(t - t')/M]. \tag{66}
\]

This approximation neglects the wave packet spreading and remains good to the first order in \( \delta p \) if \( p_1 \) is replaced by \( p_1 + \delta p \).

One can then combine \( \tilde{S} \) matrix elements and \( G \) propagators for the various elements of an interferometer to derive the interference signal. For example, we can use Eq. (64) and the previous propagator to recover formulas (36) and (40), which apply to the case of a single interaction zone (Fig. 6):

\[
\Psi_{11}(r, t) = \int d^3r_1 d^3r_0 G_{b\beta}(r, r_1, t - t_1)\tilde{S}_{ba}(r_1, t_1)G_{a\beta}(r_1, r_0, t_1 - t_0)a(r_0, t_0)
\]

\[
= \int d^3r_1 d^3r_0 \exp\{-i[E_a - \sum k(x, y, z)]M_{ba} \cdot \delta[r - r_1 - (p + \delta k)(t - t_1)/M] \exp[-i(E_a - p^2/2M)(t - t_1)/\hbar]\delta[r_1 - r_0 - p(t_1 - t_0)/M]a_0(r_0, t_0) \tag{67}
\]

with \( k' = k + \delta k \) and \( a_0(r, t) = a_0 \exp\{i(p \cdot r - (E_a + p^2/(2M))t)/\hbar\}\).

The sensitivity of atomic interferometers to external fields can be estimated with general expressions for the phase difference obtained by the same formalism. If we denote by \( \Psi(0)(t) \) the solution in the absence of interaction from \( t_0 \) to \( t \), Eq. (48) becomes \( \Psi(t) = \Psi(0)(t) \) and we may rewrite Eq. (48) as

\[
|\tilde{\Psi}(t)\rangle = T \exp\{-\frac{i}{\hbar} \int_{t_0}^{t} dt' \tilde{V}[r_{op}(t', t_1), t]\} |\Psi(0)(t)\rangle \tag{68}
\]
FIG. 6. Treatment of a single beam splitter by the $\hat{S}$ matrix method.

implying that Eqs. (50) and (51) can be used, quite generally, to calculate the phase shift introduced along an interferometer path by the perturbation $\hat{V}$.

For example, for an atom in free space, the first-order phase shift is given by the first-order $\hat{S}$ matrix:

$$\delta \varphi = -\text{Re} \int_{t_0}^{t} dt' \exp[-i \omega_{BA}(t - t')] V_{BA}^{+} [r - (p + \hbar \mathbf{k}/2)(t - t')/M, t']/\hbar$$

(69)

where $B$ and $A$ are isospinor or Pauli spinor states.

The phase shift is thus obtained by integrating the perturbation along the classical path with a recoil correction, which can be rewritten with the derivative of the perturbation taken also along the classical path:

$$\delta \varphi = -\text{Re} \int_{t_0}^{t} dt' \exp[-i \omega_{BA}(t - t')]$$

$$\left\{ V_{BA}^{+} [r - p(t - t')/M, t']/\hbar - \frac{1}{2} r_s(t, t') \cdot \nabla V_{BA}^{+} [r - p(t - t')/M, t']/\hbar \right\}$$

(70)
where
\[ r_1(t,t') = \frac{\hbar k}{M} (t - t') \] (71)
is the deviation from the unperturbed trajectory. This result generalizes that derived in Ref. 25 by a perturbation expansion of the action integral.

If \( V \) is an interatomic potential, the same formula gives the index of refraction of a gas for the matter wave \([26-28]\). In this case the \( S \) or \( T \) matrices are expressed in terms of the diffusion amplitudes.

The previous formalism and Eq. (69) can also be generalized to include the relativistic time-dilation factors and to Dirac spinors \( \psi(p) \) \([11, 29]\):
\[
\delta \varphi \sim -\text{Re} \int_{t_a}^{t_f} dt' \exp\left[ -i \omega_{BA} \sqrt{1 - \frac{v^2}{c^2}}(t - t') \right] \\
v^+_B(p + \hbar k)\psi^+_{BA}[\mathbf{r} - (p + \hbar k/2)\sqrt{1 - \frac{v^2}{c^2}}(t - t')/M,t']\psi_A(p)/\hbar
\] (72)
with the normalization condition \( \psi^+ \psi = 1 \).

II. Architecture of Interferometers

A. THE TWO-ZONE RAMSEY INTERFEROMETER

The simplest configuration for an interferometer would be the Ramsey separated fields method using only two field zones \([30]\). Using the previous rules to derive the phase shift for two field zones at coordinates \( x_1 \) and \( x_2 \), one finds the following phase factor (Fig. 7):
\[
\exp\left[ -i \delta k_x(x_2 - x_1) \right] \exp\left[ i(\varphi_1 - \varphi_2) \right] \\
= \exp\left[ -i(\Delta - \delta + k_x v_x)(x_2 - x_1)/v_x \right] \exp\left[ -ik_x(x_2 - x_1) \right] \exp\left[ i(\varphi_1 - \varphi_2) \right].
\] (73)
With this approach, in which the atomic motion is quantized, the origin of the Ramsey fringes lies in the difference \( \delta k_x \) in momentum along each arm induced by the inelasticity in the interaction and not from a comparison between an atomic clock traveling along a classical trajectory and an electromagnetic clock in two places.

However, because of recoil, this interferometer cannot be closed in space and the fringes result from only a partial transverse overlap of the wave packets. As a result, these fringes wash out because of the transverse velocity distribution \( F(v_x) \). The limit case of a flat interferometer is obtained with two-photon excitation with counterpropagating waves in each zone and an effective \( k_z = 0 \). We note that, even in this case, the fringes result from the change \( \delta k_x \) in momentum along one arm. This is to be contrasted with the case of pulsed temporal excitation by
PHASE-SHIFT CALCULATION FOR A PAIR OF FIELD ZONES

\[ \exp \left[ -i \omega t - i \vec{k} \cdot \vec{r} \right] \]

\[ + i \delta k_x (x - x_z) - i \varphi_2 \]

\[ \exp \left[ - \frac{i}{\hbar} \left( E_a - \frac{p^2}{2M} \right) T_i \right] \]

\[ \exp \left[ -i k_a (x - v_s T_i - x_1) - i \varphi_1 \right] \]

\[ a(\vec{r} - \vec{p} T_i / M) \]

\[ a(\vec{r} - (\vec{p} - h \vec{k}) T_i / M) \]

\[ - \longrightarrow \exp \left[ -i \delta k_x (x_2 - x_1) \right] \exp \left[ i(\varphi_1 - \varphi_2) \right] \]

\[ \exp \left[ - i \Delta - \delta + k_z v_x d / v_x \right] \exp \left[ - i k_x d \right] \exp \left[ i(\varphi_1 - \varphi_2) \right] \]

Fig. 7. Phase-shift calculation for a pair of field zones at coordinates \( x_1 \) and \( x_2 \) with \( x_2 - x_1 = d \): Only phase factors are displayed on the figure. To each field zone corresponds an \( S \) matrix with the laser carrier and a phase factor introduced by the correction momentum. Between field zones, we have phase factors corresponding to the action integral for free propagation. If a common space-time point \( (r,t) \) is chosen after the second field zone, the two paths originate from different space-time points before the first field zone.

plane waves, for which the fringes arise from phase factors corresponding to a change of energy, without frequency-dependent momentum change \( \delta k_x \).

B. MULTIPLE-ZONE INTERFEROMETERS

If we want to close the interferometer on itself, we need to deflect the paths in the intermediate region (Fig. 8) and, in general, use four field zones [6, 16, 17, 31, 32]. In some cases, the two opposite wave vector components of the same field can be used to deflect both paths in the middle region and the number of zones can be reduced to three [9, 33].

The four traveling waves can propagate in the same direction or one can use two counterpropagating pairs of copropagating waves [7]. In the latter case, there should be a small angle \( \theta \) between the two pairs to satisfy the Bragg condition at resonance, and the phase factor is found to be [20]

\[ \exp[2i(\Delta - \delta)d/v_s] \exp[2ikd \sin(\theta/2)] \exp[i(\varphi_4 - \varphi_3 + \varphi_2 - \varphi_1)] \quad (74) \]
INTERFEROMETER GEOMETRIES

A. SYMMETRICAL MACH-ZEHNDER

B. SKEW-SYMMETRIC

C. TRAPEZOIDAL

(a)

Fig. 8. Multiple zones interferometers. (a) Interferometer geometries, which include the neutron interferometers [1] and atom interferometers using microfabricated grids [2] as well as those using laser beams for the atomic splitters. Atomic interferometers comprising three or four copropagating traveling waves have been demonstrated in space-time with temporal pulses by Kasevich and Chu [9] and in space with spatial field zones by Morinaga and Ohuchi [59]. Atomic interferometers comprising two counterpropagating pairs of copropagating traveling waves have been demonstrated in Refs. 8, 16, and 31. The phase shifts for all these interferometers have been calculated in Ref. 7.

where $d$ is the common distance between the last two and first two field zones.

Let us emphasize again that, to introduce the recoil shift $\delta$, it is necessary to quantize the atomic motion and that, in this case, the first term arises only from a different momentum along two paths in space and hence cannot be explained outside the context of atom interferometry in a consistent way.

The first demonstration of a molecular interferometer has been carried out using an $I_2$ beam [20] by varying either the detuning $\Delta$ or the angle $\theta$, in conditions where the recoil $\delta$ is measurable (Fig. 9) [34].
1. Recoil Shift and Atomic-Mass Measurements

The $\mp$ sign of the recoil shift corresponds to two possible interferometers, which differ by internal state labels in the middle zone and which arm the intermediate deflections are along.

These two fringe systems have been observed for the first time with a calcium beam, in an interferometer configuration using three standing waves [35]. In a cell, saturation spectra result from the superposition of interferometers with all possible values for the distance $d$, letting only the central fringe survive the corresponding average. The two sets of interferometers then give rise to the recoil splitting first observed in methane at 3.39 $\mu$m [36]. It was suggested, at that time, that this splitting could be used for a precise frequency measurement of $\hbar/M$ for atomic systems. With the advent of cold atoms this possibility was beautifully illustrated by Chu and coworkers [37], who demonstrated an accuracy in the range of $10^{-7}$ in the case of cesium. For this purpose, new interferometer geometries have been designed to increase the recoil shift thanks to an increase of the deflection of one arm using multiple intermediate $\pi$ pulses (Fig. 10) [38].
III. Sensitivity to Gravitational and Electromagnetic Fields: A Unified Approach through the Dirac Equation

To describe consistently and rigorously the various phase shifts resulting from interactions with external electromagnetic or gravitational fields, it is necessary to introduce a framework in which the motion of the center-of-mass of atoms is treated relativistically. This is possible because one can associate a relativistic quantum field with each internal state of the atoms [11]. In other words, an atom in a given internal state is considered an elementary particle with a mass corre-
FIG. 10. (b) Corresponding diagrams for an interferometer that starts with a coherent superposition of states $|a,0\rangle$ and $|a',0\rangle$, created without momentum exchange, e.g., by a Doppler-free two-photon interaction or by a Raman process with copropagating beams. States $a$ and $a'$ may be either different energy states (e.g., $1S$ and $2S$ states of hydrogen) or different Zeeman sublevels of the same energy state (e.g., $M_F = \pm 1$ sublevels of the $2S$ ($F = 1$) state of hydrogen). With a proper choice of frequencies (or polarizations), recoil momentum is transferred only to the path starting with state $|a',0\rangle$. (Reprinted from Laser Spectroscopy, Int. Conf. 1994, p. 77, with permission of the publisher, American Institute of Physics.)

sponding to its rest energy and a spin equal to the total angular momentum of the state. An atomic state having $F = 0, \frac{1}{2}, 1, \ldots$ corresponds, respectively, to a Klein–Gordon field, a Dirac field, a Proca field, \ldots.

A. COVARIANT DIRAC EQUATIONS

As an example that avoids the complications and difficulties of higher spins but still retains the richness introduced by angular momentum, we shall consider the case of two-level atoms with $F = \frac{1}{2}$ in each level. The interaction with the electromagnetic field $F_{\alpha\beta}$ is introduced via off-diagonal electric-dipole and diagonal magnetic-dipole terms in the Lagrangian. Inertial and gravitational fields can be introduced in two ways: first, via the Dirac equation in curved space–time [39]; second, as a second-rank tensor field in flat space–time [40].
1. Curved Space-Time Approach

In the first approach, the following coupled Dirac equations are written for each level\(^3\) (with \(\epsilon_{0123} = 1\)):

\[
\begin{align*}
\gamma^a \gamma^b \partial_\mu \psi_a - M_B \gamma^b \psi_a - (i \mu_\alpha / 2c) F_{\alpha \beta} \gamma^\alpha \gamma^b \psi_a - (i \mu_\beta / 4) \tilde{F}_{\alpha \beta} \gamma^\alpha \gamma^b \psi_a &= 0 \\
\gamma^a \gamma^b \partial_\mu \psi_b - M_B \gamma^b \psi_b - (i \mu_\alpha / 2c) F_{\alpha \beta} \gamma^\alpha \gamma^b \psi_b - (i \mu_\beta / 4) \tilde{F}_{\alpha \beta} \gamma^\alpha \gamma^b \psi_b &= 0
\end{align*}
\]

(75)

where \(e^a_\mu\) is a tetrad or vierbein that defines a local inertial coordinate system in which a spinor can be introduced \([41]\) and \(\Gamma_\mu = \frac{1}{8} [\gamma^\alpha, \gamma^\beta] e^\alpha_\mu e^\beta_\nu\) is a spinorial connection. The tetrad field is obtained from the metric tensor by the condition:

\[
e^a_\mu e^\mu_\nu = \eta_{\alpha \beta} = \text{diag}(+ - - -).
\]

(76)

As pointed out in Ref. 39, these equations can be cast together as a single equation for an eight-component isospinor and one can introduce a generalized covariant derivative and a generalized connection. Quite generally, a matter-wave interferometer can be viewed as a device to detect this connection.

2. Case of Inertial Fields

As an example, for the acceleration \(a\) and the rotation \(\Omega\), the tetrads can be written as \([42]\)

\[
e^0_\mu = \frac{1}{1 + a \cdot x / c^2}; \quad e^i_\mu = \frac{[\Omega \times x]^i / c}{1 + a \cdot x / c^2}; \quad e^0_\mu = 0; \quad e^i_\mu = 1
\]

(77)

and the coupled Dirac equations become

\[
\begin{align*}
\gamma^a \gamma^b \partial_\mu \psi_a &= \gamma^0 M_a c^2 (1 + a \cdot x / c^2) + c \alpha \cdot p + [(a \cdot x)(\alpha \cdot p)] \\
&+ (\alpha \cdot p)(a \cdot x)/2c - \Omega \cdot (L + S)] \psi_a + (i \mu_\alpha / 2c) \gamma^a \gamma^0 \gamma^\beta F_{\alpha \beta} \psi_a \\
&+ (i \mu_\beta / 4) \gamma^0 \gamma^\alpha \gamma^\beta \tilde{F}_{\alpha \beta} \psi_a \\
\gamma^a \gamma^b \partial_\mu \psi_b &= \gamma^0 M_a c^2 (1 + a \cdot x / c^2) + c \alpha \cdot p + [(a \cdot x)(\alpha \cdot p)] \\
&+ (\alpha \cdot p)(a \cdot x)/2c - \Omega \cdot (L + S)] \psi_b + (i \mu_\alpha / 2c) \gamma^a \gamma^0 \gamma^\beta F_{\alpha \beta} \psi_b \\
&+ (i \mu_\beta / 4) \gamma^0 \gamma^\alpha \gamma^\beta \tilde{F}_{\alpha \beta} \psi_b
\end{align*}
\]

(78)

where \(L\) is the orbital angular momentum operator and \(S = \hbar \Sigma / 2\). These equations display clearly the red-shift effects, the kinetic energy term, the Sagnac ef-

\(^3\)For particles without internal states, the internal state labels \(a\) and \(b\) can be replaced by the channel labels I and II to designate two resolved momentum states. For charged particles (electrons or atomic ions), one should add the interaction with the four-vector potential \(A_\mu\) through the minimal coupling \(\partial_\mu \rightarrow \partial_\mu - i(\gamma^0 / \hbar) A_\mu\).
fect, and the spin-rotation coupling. In their nonrelativistic limit, one obtains the Thomas precession Hamiltonian term as

\[
\frac{1}{2M_0c^2} \mathbf{S} \cdot (\mathbf{a} \times \mathbf{p}).
\] (79)

3. Weak Gravitational Fields

For more general weak gravitational fields, it is usual to introduce a quasi-Minkowskian coordinate system and to write the metric tensor as

\[
g_{\mu \nu} = \eta_{\mu \nu} + h_{\mu \nu}
\] (80)

where \(h_{\mu \nu}\) is a small departure \(|h_{\mu \nu}| \ll 1\) from the Minkowski metric (linearized theory of gravity). This approximation holds very well in most cases. Various choices can be made for the tetrads calculated from conditions (76). In Ref. 39 the following choice was made:

\[
e_\mu^0 = \delta_\mu^0(1 - h_{00}/2); \quad e_\mu^i = \delta_\mu^i - \frac{1}{2} h_{0i} - \frac{1}{2} \delta_\mu^0 h_0^i.
\] (81)

Another choice consistent with Eq. (77) would be

\[
e_\mu^0 = \delta_\mu^0 - \frac{1}{2} \delta_\mu^0 h_{00} - \delta_\mu^k h_0^k \quad e_\mu^i = \delta_\mu^i - \frac{1}{2} \delta_\mu^k h_i^k.
\] (82)

Finally, a natural and covariant choice is

\[
e_\mu^a = \delta_\mu^a - \frac{1}{2} h_{ab} \quad (83)
\]

with which the coupled Dirac equations become

\[
i\hbar \gamma^\mu \partial_\mu \psi_a - M_a c \psi_a - (i\hbar/2)h^{\mu \nu} \gamma^\mu \partial_\nu \psi_a - (i\hbar/4)\left(\partial_\mu h^{\mu \nu}\right) \gamma^\nu \psi_a
\]

\[
+ (i\hbar/4) \left(\partial_\mu h_0^\mu\right) \gamma^\mu \psi_a - (i\mu_b/2c) \gamma^\mu \gamma^\beta F_{\alpha \beta} \psi_a - (i\mu_b^*/4) \gamma^\nu \gamma^\beta \tilde{F}_{\alpha \beta} \psi_a = 0.
\]

\[
i\hbar \gamma^\mu \partial_\mu \psi_a - M_a c \psi_a - (i\hbar/2)h^{\mu \nu} \gamma^\mu \partial_\nu \psi_a - (i\hbar/4)\left(\partial_\mu h^{\mu \nu}\right) \gamma^\nu \psi_a
\]

\[
+ (i\hbar/4) \left(\partial_\mu h_0^\mu\right) \gamma^\mu \psi_a - (i\mu_a/2c) \gamma^\mu \gamma^\beta F_{\alpha \beta} \psi_a - (i\mu_a/4) \gamma^\nu \gamma^\beta \tilde{F}_{\alpha \beta} \psi_a = 0.
\] (84)

4. Flat Space-Time Approach

It is remarkable that the same equations are obtained within the framework of the tensor-field theory of gravity in flat space and time with the Lagrangian

\[
L = (i\hbar/2) \bar{\psi}_a \gamma^\mu \partial_\mu \psi_a - (i\hbar/2) \bar{\psi}_a \gamma^\mu \psi_a \gamma^\nu \psi_a - M_a c \bar{\psi}_a \psi_a - (1/2c) \hbar h_{\mu \nu} T_{\mu \nu}
\]

\[
- (i\mu_d/2c) F_{\mu \alpha} \psi_a \gamma^\mu \gamma^\alpha \bar{\psi}_a - (i\mu_a/4) \tilde{F}_{\mu \nu} \bar{\psi}_a \gamma_{\mu \nu} \psi_a + (a \leftrightarrow b)
\] (85)
where the coupling with the tensor field $h_{\mu \nu}$ describing gravitation is written in the most natural way with the atomic energy–momentum tensor:

$$T_{\mu \nu} = (i\hbar c/4)(\bar{\psi}_a \gamma^\mu \gamma^\nu \psi_a + \bar{\psi}_a \gamma^\mu \gamma^\nu \psi_a - \gamma^\mu \bar{\psi}_a \gamma^\nu \psi_a - \gamma^\mu \bar{\psi}_a \gamma^\nu \psi_a)$$

$$- \gamma^\mu \{[i\hbar c/2] [\gamma_a \gamma^\mu \gamma^\nu \psi_a - \gamma_a \gamma^\nu \psi_a] - Ma c^2 \gamma^\mu \psi_a \}.$$  

(86)

Therefore, it is also possible to give a description of gravitational effects in atom interferometry, using quantum field theory in flat space–time and the scattering matrix for the one-graviton exchange between the atom and an external source.

5. Equations for the "Renormalized Spinor"

The difficulty that we meet with the previous equations is that the atomic density probability involves the field $h_{\mu \nu}$ and a "renormalization" of the spinor is required. We define new spinors

$$\theta_{a, b} = (1 - h_{00}/4 + h_{\rho}^0/4 - h_{ij}^0 \gamma_0 \gamma^j/4)\psi_{a, b}$$

(87)

such that the probability density is

$$J_{a/c} = \theta^*_a \theta_a + \theta^*_b \theta_b.$$  

(88)

The evolution of these spinors is governed by the equation (39)

$$i\hbar \partial_t \theta_a = -i\hbar c \gamma^\rho \gamma^i \partial_j \theta_a + Ma c^2 \gamma^0 \theta_a - (c/4)\{p_j \gamma^0 \gamma^j \theta_a + M_a c^2 \gamma^0 \theta_a\}$$

$$+ (i\hbar c/8)(\partial_t h_{00} - \partial_j h_{0j} \gamma^j \gamma^i \theta_a + (c/2)\{p_j \gamma^0 \gamma^j \theta_a + (c/4)\{p_j \gamma_0 \gamma^0 \gamma^j \theta_a + (i\mu_j)2 \gamma^\rho \gamma^\alpha \gamma^0 F_{\alpha \beta} \theta_a + (i\mu_0)2 \gamma^\rho \gamma^0 \gamma^0 F_{\alpha \beta} \theta_a$$

(89)

with $p_i = i\hbar \partial_i$ and a similar equation for $\theta_b$ in which $(a \leftrightarrow b)$.

B. GRAVITATIONAL AND ELECTROMAGNETIC PHASE SHIFTS

Equation (89) displays all the terms that may lead to a phase shift in an interferometer:

- The kinetic energy term gives rise to the recoil splitting already discussed.
- The terms involving $h_{00}$ lead to the gravitational shift, to shifts involving higher derivatives of the gravitational potential [44], and to the Thomas precession. Among these, the gravitational shift $-k \cdot gT^2$ [7] first observed with neutrons in the COW experiments [45], has been measured using atom interferome-

4For the gravitational terms, which do not involve the spin and the internal structure of the atoms, Eq. (72) gives the same first-order phase shifts as the Linet–Tourenec formula [43].
try with an accuracy of $3.10^{-8}$ by Kasevich and Chu [9] using a temporal sequence of three Raman pulses. As noted in Ref. 7 the absence of the mass in the phase shift expression is a consequence of the equivalence principle, which is also applicable to the neutron interferometer. Precision gravimeters are presently under construction.

- The next term, which involves $h_0$, describes the Sagnac effect, with

$$h = \{h_{0k}\} = \Omega \times x/c$$

and leads to a shift equal to

$$4\pi A \cdot \Omega / (\lambda_{db}v).$$

It was first demonstrated for atomic waves by Riehle and coworkers (Fig. 11) [8]. For identical areas $|A|$, the effect is $Mc^2/h\nu$ times larger than that for photons. Large atom interferometers being developed should have a sensitivity of $10^{-11}$ rad/s/$\sqrt{Hz}$ [46], which might be sufficient to detect the Lense-Thirring effect. This is a general relativity effect, specifically a dragging of the metric by a nearby rotating mass that induces a rotation on the order of $10^{-14}$ rad/s in the case of the earth.

- The next term, also due to rotation known as the \textit{spin-rotation effect}, gives a shift equal to $-2\Omega \cdot ST$. This effect is used in magnetic resonance gyroscopes [47] and should be accessible using atom interferometry if, for example, one changes the spin on one of the two coincident paths of a Doppler-free two-photon, two-zone interferometer.

- The seventh term describes genuine general relativity effects and involves $h_i$ components, which exist, for example, in the Schwarzschild metric produced by a massive spherical body or which are produced by a gravitational wave. In the case of the Schwarzschild metric, this term would induce the de Sitter precession (in combination with the Thomas precession quoted previously) [39]. The Hamiltonian term created by a gravitational wave is very similar to the kinetic energy term and the corresponding shift has an expression comparable to the recoil shift [29]:

$$\pm h(h \omega^2/Mc^2)T \sim \pm h(\lambda_{db}/\lambda_{opt})(d/\lambda_{opt})$$

where $h$ is the amplitude of the gravitational wave. In addition, there is also a very small spin-gravitation effect [39], owing to the coupling between the spin and the space–time curvature.

Finally, the coupling terms with the electromagnetic field describe the following:

- The diagonal magnetic dipole interaction is responsible for the Zeeman effect and its motional counterpart gives the Aharonov–Casher effect in a static
Fig. 11. Sagnac effect for atomic waves (from [8]). (a) Experimental setup: Typical atom interferometer using two counterpropagating pairs of copropagating laser waves to split, deflect, and recombine the calcium beam from an oven at $T \approx 1000$ K. The dye laser is tuned across the resonance with the intercombination transition $^3P_1 - ^1S_0$ of $^{40}$Ca at 657.46 nm. The whole apparatus can be rotated at the angular velocity $\Omega$. (b) Interference signal obtained by monitoring the fluorescence from the excited state: Curves $b$ and $d$ correspond to opposite rotation directions at $\Omega = \pm 0.09$ sec$^{-1}$ and the three others to $\Omega = 0$. The two central fringes are separated by a recoil splitting of 23.1 kHz.
electric field [48], for which a very nice demonstration has been given in Ref. 49.
• The off-diagonal electric dipole interaction is responsible for light shifts [50] (and also for the beam splitting).

In the case of electrons or ions, one should not forget the minimal coupling term that gives rise to the Aharonov–Bohm effect [51].

IV. Conclusions and Directions of Future Progress

We have seen that, to a large extent, we can unify the description of matter–wave interferometers: the description of the beam splitters can be given in terms common to particles with and without internal structure. Particles with internal structure, such as atoms or molecules, offer the additional interesting possibility to be excited inelastically by the splitter. The use of quantum field theory provides another ground for a unified framework of all particles used in interferometry, from photons to the most massive ones and from bosons to fermions, all of which are considered interacting fields. Furthermore, one can write these interactions in a covariant way and thus introduce the relativistic effects in the most natural and consistent way. This scheme has also the advantage that it underlines similarities and differences between the various interactions; for example, some gravitational effects can be presented as gravitomagnetic or gravitoelectric using the four-vector potential $h_{00}h_{0\gamma}$, but clearly this analogy ig-
nores the other components of the tensor \( h_{\mu \nu} \). Much work is still to be done (es-
pecially on coordinate systems) to get a clear and full picture of all possible rela-
tivistic effects and to conclude whether or not they are detectable by a given in-
terferometer, but we hope that the present approach can be of some help in this

direction.

Among the possibilities of improving seriously the sensitivity of atom inter-
ferometers, we may speculate on these:

1. Methods for slowing molecular beams or improved beam splitters using
higher laser frequencies (UV, VUV) with good optical quality allowing one to
use thermal molecular beams at room temperature; new interaction schemes
such as the adiabatic fast passage [7, 52, 53].

2. More generally, better atom optics, such as high-quality mirrors, magnetic
mirrors, magnetic guiding of the atoms, and atom fibers.

3. Better interferometer architecture, such as multiple \( \pi \) pulses and figure 8
interferometers to detect the Lense–Thirring effect.

4. Coherent sources of atoms. Presently the minimum phase shift that can be
detected is related to the number of atoms by [54]:

\[
\delta \varphi \approx \frac{1}{\sqrt{N_{\text{atoms}}}}
\]

(93)

and the path difference is limited by the coherence of atomic beams. Clearly, it
would be interesting to use either the Bose–Einstein condensates recently
demonstrated [55] or future atomasers [56, 57] as new sources of atoms for atom
interferometry. It has been suggested [58] that the previous sensitivity might be
replaced by

\[
\delta \varphi \approx \frac{1}{N_{\text{atoms}}}.
\]

(94)

In the future, active atomic systems should offer a large gain in sensitivity to
external fields; for example, one could imagine building a ring atomic wave

gyro. Finally, the recent success of atom interferometers in detecting the index
of refraction of an atomic cloud [26–28] suggests the following experiments:

1. One could explore the index of refraction of a cloud of excited atoms, to
demonstrate stimulated emission of atoms. The formula for the index is obtained
by exchanging real and imaginary parts in the formula for the gain in Refs. 56
and 57.

2. One could use an atom interferometer in the same way to detect Bose–
Einstein condensation in a cold cloud of atoms and to investigate interactions
and correlations in the condensate.

3. Beyond the index for spin zero or depolarized particles, one could think of
probing the index properties of atomic media for polarized atomic waves with
higher spins, generalizing the usual effects of birefringence and dichroism for optical waves. Optically active molecules should exhibit rotatory power and circular dichroism also for polarized atomic waves, through the chiral part of the intermolecular potential; and this could be explored, for example, in a Stern–Gerlach interferometer [10] crossing a cloud of such molecules. It should also be noted that the possibility to use chiral molecules in the molecular interferometer itself, leads to interesting new tests of parity violation; for example, one could test the equivalence principle against parity in a COW type experiment.

In these cases, generalized interferometers, with different internal states in both arms, should be useful tools.

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