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Quantum Theory of Atom-Wave Beam Splitters and Application to Multidimensional Atomic Gravito-Inertial Sensors

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We review the theory of atom-wave beam splitters using atomic transitions induced by electromagnetic interactions. Both the spatial and temporal dependences of the e.m.[3] fields are introduced in order to compare the differences in momentum transfer which occur for pulses either in the time or in the space domains. The phases imprinted on the matter-wave by the splitters are calculated in the limit of weak e.m. and gravitational fields and simple rules are derived for practical atom interferometers. The framework is applicable to the Lamb-Dicke regime. Finally, a generalization of present 1D beam splitters to 2D or 3D is considered and leads to a new concept of multidimensional atom interferometers to probe inertial and gravitational fields especially well-suited for space experiments.

KEY WORDS: Gravito-inertial sensor; atom-wave beam splitter.

1. INTRODUCTION

A very convenient beam splitter for atom waves, easily and accurately controlled, is realized through the interaction of atoms with resonant laser beams and more generally resonant e.m. waves [1]. This interaction leads to the absorption of both the energy and the momentum of an effective photon in a one-photon or multiphoton process such as a Raman process [2–6, 27]. It was demonstrated

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[3] e.m. = electromagnetic

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recently [7–10] that the main contribution to the phase shift in atom interferometers comes from the phase imprinted on the matter-wave by the beam splitters (see Appendix C). A good understanding of this phase is thus essential to give a proper description of atom interferometry. Many papers have been devoted already to the theory of beam splitters covering various aspects [11–17]. The present paper has essentially a tutorial ambition but tries also to answer some specific questions and to suggest some new directions for the future. For example, it was recognized that e.m. pulses in the time domain (separated in time) and pulses in the space domain (spatially separated) have a different action on an extended atom wave and lead to different expressions for the phase shift. To understand these differences, it is necessary to give a quantum description of the splitting process without assuming any classical point of intersection where the interaction takes place. To keep easily tractable expressions and focus on the previous point, we have limited ourselves to a first-order theory leaving the strong-field case for a future publication [18]. In this limit, we derive the *ttt* theorem, which gives simple expressions for the phase shift introduced by the beam splitter.

In Appendix A, a Schrödinger-type equation valid for both massive and non-massive particles is briefly rederived from the Klein-Gordon equation in curved space-time. Appendix B is a short reminder on the ABCD matrices used to write the propagators of atom waves and, in Appendix C, we recall the general formula for the phase difference in atom interferometers. In each of these last two appendices, we give the example of the action of a gravitational wave as an illustration. The calculation of the first-order scattered amplitude, in a one-photon process, is detailed in Appendix D. Finally, in Appendix E, we show how to extend this result to two-photon transitions and derive the corresponding recoil corrections.

The simple model of 1D atom beam splitters provided by this weak-field approach is a first basis to understand the principles of 2 or 3D atom beam splitters in which atom waves are diffracted from an initial atomic cloud in orthogonal directions of space. With such splitters one could build a coherent superposition of atomic clouds, images of the initial cloud and forming a macroscopic 3D figure in space, such as a trihedron, a cube, an octahedron or an extended grating, expanding or at rest. This macroscopic quantum superposition would be an ideal inertial reference system that could be used to probe simultaneously several components of the gravitational field through an interference with itself at a later time. Such possibilities are clearly offered today by ultra-cold atomic clouds, Bose-Einstein condensates or atom lasers for future space experiments.

2. SCHRÖDINGER EQUATION AND INTERACTION HAMILTONIAN

We start with the Schrödinger equation in gravitational and inertial fields (see Appendix A and references [7, 8]):

$$i\hbar\frac{\partial|\Psi(t)\rangle}{\partial t} = \left[H_0 + \frac{1}{2M}\vec{p}_{\text{op}}\cdot \overset{\Rightarrow}{\beta}(t)\cdot\vec{p}_{\text{op}} - \vec{\Omega}(t)\cdot(\vec{L}_{\text{op}}+\vec{S}_{\text{op}}) \right.$$
$$\left. -M\vec{g}(t)\cdot\vec{r}_{\text{op}} - \frac{M}{2}\vec{r}_{\text{op}}\cdot \overset{\Rightarrow}{\gamma}(t)\cdot\vec{r}_{\text{op}} + V(\vec{r}_{\text{op}},t)\right]|\Psi(t)\rangle, \quad (1)$$

where H_0 is an internal Hamiltonian of the atom with eigenvalues E_a, E_b.., where $V(\vec{r}_{\text{op}}, t)$ is the electric or magnetic dipole interaction Hamiltonian with the electromagnetic field in the beam splitters and where the other terms contribute to a general external motion Hamiltonian[4] in the presence of various gravito-inertial fields including a rotation term (with angular velocity $\vec{\Omega}(t)$), a gravity field $\vec{g}(t)$ and its gradients $\overset{\Rightarrow}{\gamma}(t)$ and possibly other contributions coming from the metric tensor in $\overset{\Rightarrow}{\beta}(t)$ (representing for example the effect of gravitational waves in a given gauge...). We have used the usual Dirac bra and ket notation in which $\vec{r}_{\text{op}}$, $\vec{p}_{\text{op}}$, $\vec{L}_{\text{op}}$ and $\vec{S}_{\text{op}}$ are respectively the position, linear momentum, angular momentum and spin operators.

A series of unitary transformations:

$$|\tilde{\Psi}(t)\rangle = U_0^{-1}(t,t_1)|\Psi(t)\rangle, \quad (2)$$

where t_1 is an arbitrary time (which will disappear from the final result) and where ($\mathcal{T}$ is a time-ordering operator)

$$U_0(t,t_1) = U_E(t,t_1)e^{-iH_0(t-t_1)/\hbar}\mathcal{T}\exp\left[\frac{i}{\hbar}\int_{t_1}^{t}\vec{\Omega}(t')\cdot\vec{S}_{\text{op}}dt'\right] \quad (3)$$

$$U_E(t,t_1) = U_R(t,t_1)U_1(t,t_1)\dots U_6(t,t_1) \quad (4)$$

(see Appendix 2 of [8]), eliminates one term after the other and brings the Schrödinger equation to the simple form:

$$i\hbar\frac{\partial|\tilde{\Psi}(t)\rangle}{\partial t} = \tilde{V}(\vec{r}_{\text{op}},\vec{p}_{\text{op}},t)|\tilde{\Psi}(t)\rangle, \quad (5)$$

with

$$\tilde{V}(\vec{r}_{\text{op}},\vec{p}_{\text{op}},t) = \hat{V}(\vec{R}_{\text{op}}(t,t_1),t) \quad (6)$$

($\hat{V} = e^{iH_0(t-t_1)/\hbar}Ve^{-iH_0(t-t_1)/\hbar}$ in the absence of spin-rotation interaction) and

$$\vec{R}_{\text{op}}(t,t_1) = U_E^{-1}(t,t_1)\vec{r}_{\text{op}}U_E(t,t_1) \quad (7)$$

$$= A(t,t_1)\cdot\vec{r}_{\text{op}} + B(t,t_1)\cdot\vec{p}_{\text{op}}/M + \vec{\xi}(t,t_1) \quad (8)$$

[4] This means relative to the motion of the center of mass. If this motion is relativistic, M should be replaced by M^* as discussed in Appendix A.

which reduces to $\vec{r}_{\rm op} + \vec{p}_{\rm op}(t - t_1)/M + \vec{\xi}(t, t_1)$ in the absence of rotation and field gradient. In the general case, the $ABCD$ matrices and the vector $\vec{\xi}$ are given in Appendix B.

The solution of the Schrödinger equation is

$$|\Psi(t)\rangle = U_0(t, t_1)\mathcal{T}\exp\left[\frac{1}{i\hbar}\int_{t_o}^{t} dt'\hat{V}(\vec{R}_{\rm op}(t', t_1), t')\right]U_0(t_1, t_0)|\Psi(t_0)\rangle. \quad (9)$$

In the position representation

$$\mathcal{K}_\alpha(\vec{r}, \vec{r}_1, t, t_1) = \langle\vec{r}, \alpha|U_0(t, t_1)|\vec{r}_1, \alpha\rangle \quad (10)$$

is the propagator of state α in the absence of laser field and

$$\begin{aligned}\alpha(\vec{r}, t) &= \langle\vec{r}, \alpha|\Psi(t)\rangle \\ &= \langle\vec{r}, \alpha|U_0(t, t_1)|\Psi(t_1)\rangle \\ &= e^{iS_\alpha(t,t_1)/\hbar}e^{i\vec{p}_\alpha(t)\cdot(\vec{r}-\vec{r}_c(t))/\hbar}\mathcal{F}(\vec{r} - \vec{r}_c(t), X(t), Y(t)), \quad (11)\end{aligned}$$

where the action $S_\alpha(t, t_1)$, the momentum $\vec{p}_\alpha$, the wave-packet center position $\vec{r}_c(t)$ and the widths matrices $X(t), Y(t)$ are given by the ABCDξ theorem for 3D Hermite-Gauss envelopes $\mathcal{F}$ [8]. The time-ordered exponential $\mathcal{T}\exp\left[\frac{1}{i\hbar}\int_{t_o}^{t} dt'\hat{V}(\vec{R}_{\rm op}(t', t_1), t')\right]$ is a transition operator between internal states α, that we shall evaluate now.

For one-photon transitions in a two-level system, the matrix element of the Hamiltonian of interaction with the e.m. waves is[5]

$$V_{ba}(\vec{r}, t) = -\sum_{\pm}\hbar\Omega_{ba}^{\pm}e^{i(\omega t\mp kz+\varphi^\pm)}F(t - t_A)U_0^\pm(\vec{r} - \vec{r}_A) + c.c. \quad (12)$$

where Ω_{ba} is a Rabi frequency, where

$$U^\pm(\vec{r}) = \frac{w_0^2}{4\pi}\int d^3k \exp\left[-\frac{\left(k_x^2 + k_y^2\right)w_0^2}{4}\right] e^{i(k_x x+k_y y+k_z z)}\delta\left(k_z \pm k \mp \frac{k_x^2 + k_y^2}{2k}\right) \quad (13)$$

$$= \frac{w_0^2}{4\pi}\int dk_x dk_y \exp\left[-\frac{\left(k_x^2 + k_y^2\right)w_0^2}{4}\left(1 \mp i\frac{2}{kw_0^2}z\right)\right] e^{i(k_x x+k_y y)}e^{\mp ikz} \quad (14)$$

[5] For simplicity, we have not introduced the dispersion $k(\omega)$ within the field envelope F.

$$= L^{\pm}(z) \exp\left[- L^{\pm}(z)(x^2 + y^2)/w_0^2\right] e^{\mp ikz} \tag{15}$$

$$= U_0^{\pm}(\vec{r}) e^{\mp ikz} \tag{16}$$

reflects the Gaussian beam geometry (see e.g. [19] for the expression of the complex Lorentzian $L^{\pm}(z)$), and where

$$F(t - t_A) = \int \frac{d\omega'}{\sqrt{2\pi}} \tilde{F}(\omega' - \omega) e^{i(\omega' - \omega)(t - t_A)} \tag{17}$$

is a temporal envelope. Thus the Fourier representation of the interaction Hamiltonian matrix element is

$$\begin{aligned} V_{ba}(\vec{r}, t) &= \sum_{\pm} \int \frac{d^3k' d\omega'}{(2\pi)^2} V_{ba}^{\pm}(\vec{k}', \omega') e^{i\vec{k}' \cdot \vec{r} + i\omega' t} \\ &= -\sum_{\pm} \hbar \Omega_{ba}^{\pm} \frac{\sqrt{2\pi} w_0^2}{2} e^{i(\omega t + \varphi^{\pm})} e^{\mp ikz_A} \\ &\quad \int \frac{d^3k' d\omega'}{(2\pi)^2} \exp\left[-\frac{\left(k_x'^2 + k_y'^2\right) w_0^2}{4} \right] e^{i\vec{k}' \cdot (\vec{r} - \vec{r}_A)} \\ &\quad \delta\left(k_z' \pm k' \mp \frac{k_x'^2 + k_y'^2}{2k'} \right) \tilde{F}(\omega' - \omega) e^{i(\omega' - \omega)(t - t_A)} + c.c. \end{aligned} \tag{18}$$

with a positive and negative temporal frequency component

$$\begin{aligned} V_{ba}^{\pm +}(\vec{k}', \omega') = -\hbar \Omega_{ba}^{\pm} \frac{\sqrt{2\pi} w_0^2}{2} e^{-i(\vec{k}' \cdot \vec{r}_A + \omega' t_A)} e^{i(\omega t_A \mp kz_A + \varphi^+)} \\ \tilde{F}(\omega' - \omega) \tilde{U}_0(\vec{k}_\perp') \delta\left(k_z' \pm k \mp \frac{k_\perp'^2}{2k} \right) \end{aligned} \tag{19}$$

$$\begin{aligned} V_{ba}^{\pm -}(\vec{k}', \omega') = -\hbar \Omega_{ba}^{\pm} \frac{\sqrt{2\pi} w_0^2}{2} e^{-i(\vec{k}' \cdot \vec{r}_A + \omega' t_A)} e^{-i(\omega t_A \mp kz_A + \varphi^+)} \\ \tilde{F}(\omega' + \omega) \tilde{U}_0(\vec{k}_\perp') \delta\left(k_z' \mp k \pm \frac{k_\perp'^2}{2k} \right) \end{aligned} \tag{20}$$

(here $\tilde{F}$ and $\tilde{U}_0$ are supposed to be real and even, but this assumption is easily removed).

With the rotating-wave approximation (RWA)

$$\hat{V}(\vec{R}_{\text{op}}(t, t_1), t) = \begin{pmatrix} 0 & \mathcal{V}_{ba}^{-}(t, t_1) \\ \mathcal{V}_{ab}^{+}(t, t_1) & 0 \end{pmatrix} \tag{21}$$

and

$$|\Psi(t)\rangle = U_0(t,t_1)\mathcal{T}\exp\left[\frac{1}{i\hbar}\int_{t_0}^{t}dt'\begin{pmatrix}0 & \mathcal{V}_{ba}^{-}(t',t_1)\\ \mathcal{V}_{ab}^{+}(t',t_1) & 0\end{pmatrix}\right]U_0(t_1,t_0)|\Psi(t_0)\rangle, \tag{22}$$

where we abbreviated

$$\mathcal{V}_{ba}^{-}(t,t_1) = \sum_{\pm} V_{ba}^{\pm -}(\vec{R}_{\text{op}}(t,t_1),t)e^{i\omega_{ba}(t-t_1)} \tag{23}$$

$$\mathcal{V}_{ab}^{+}(t,t_1) = \sum_{\pm} V_{ab}^{\pm +}(\vec{R}_{\text{op}}(t,t_1),t)e^{-i\omega_{ba}(t-t_1)}. \tag{24}$$

The time-ordered exponential has been calculated in a number of cases in references [6, 11, 12] but, here, we shall rather outline the weak-field approach, which is more transparent for a tutorial.

3. FIRST-ORDER PERTURBATION THEORY AND *ttt* THEOREM

In the weak-field limit, the first-order excited state amplitude is simply related to the lower-state unperturbed amplitude by:

$$\langle b|\Psi^{(1)}(t)\rangle = \frac{1}{i\hbar}\langle b|U_0(t,t_1)|b\rangle \int_{t_o}^{t} dt' V_{ba}^{+-}(\vec{R}_{\text{op}}(t',t_1),t')e^{i\omega_{ba}(t'-t_1)} \langle a|U_0(t_1,t_0)|a\rangle\langle a|\Psi^{(0)}(t_0)\rangle \tag{25}$$

This amplitude is calculated in the position representation in Appendix D.

In the temporal beam splitter case, the excited state amplitude at $(\vec{r},t)$ is found to be

$$b^{(1)}(\vec{r},t) = M_{ba}e^{iS_b(t,t_A)/\hbar}e^{i\vec{p}_b(t)\cdot(\vec{r}-\vec{r}_c(t))/\hbar}e^{-i(\omega t_A - kz_c(t_A)+\varphi^+)} e^{iS_a(t_A,t_0)/\hbar}\mathcal{F}(\vec{r}-\vec{r}_c(t),X(t),Y(t)) \tag{26}$$

with the momentum change

$$\vec{p}_b(t_A) = \vec{p}_a(t_A) + \hbar k\hat{z}, \tag{27}$$

and where M_{ab} is a constant factor defined in Appendix D, S_α the classical action and where $\vec{r}_c(t)$, $X(t)$, $Y(t)$ are, respectively, the central position and width parameters of the atomic wave packet given by the $ABCD\xi$ law [7, 8, 19].

In the spatial beam splitter case we get

$$b^{(1)}(\vec{r},t) = M_{ba}e^{iS_b(t,t'_A)/\hbar}e^{i\vec{p}_b(t)\cdot(\vec{r}-\vec{r}_c(t))/\hbar}e^{-i(\omega t'_A - kz_c(t'_A)+\varphi^+)} e^{iS_a(t'_A,t_0)/\hbar}\mathcal{F}(\vec{r}-\vec{r}_c(t),X(t),Y(t)), \tag{28}$$

where t'_A is such that $x_c(t'_A) = x_A$. This is the same formula as in the temporal case but with the momentum change

$$\vec{p}_b(t'_A) = \vec{p}_a(t'_A) + \hbar k_{0x}\hat{x} + \hbar k\hat{z} \tag{29}$$

i.e. with an additional momentum $\delta\vec{k} = k_{0x}\hat{x} = (\Delta - kv_{0z} - \delta)\,\hat{x}/v_{0x}$ in the longitudinal direction defined by the unit vector $\hat{x}$ and proportional to the detuning[6]

This proves the ttt theorem (where ttt stands for $t_0 t_A t$), which is the basis for the calculation of exact phase shifts in atom interferometry [9, 10] (see Appendix C): *When the dispersive properties of a laser beam splitter are neglected (i.e. the wave packet shape is preserved) its effect may be summarized, besides an obvious momentum change, by the introduction of both a phase and an amplitude factor for the atom wave*

$$M_{ba}e^{-i(\omega^* t^* - \tilde{k}^* q^* + \varphi^*)} \tag{30}$$

where t^* and q^* depend on t_A and q_A, the central time and central position of the electromagnetic pulse used as an atom beam splitter[7]*: for a temporal beam splitter*

$$\begin{aligned} t^* &\equiv t_A \\ q^* &\equiv q_{cl}\,(t_A) \\ k^* &\equiv k \\ \omega^* &\equiv \omega \\ \varphi^* &\equiv \varphi \text{ (laser phase)}, \end{aligned} \tag{31}$$

and for a spatial beam splitter

$$\begin{aligned} q^* &\equiv q_A \\ t^* \text{such that } q_{cl}(t^*) &\equiv q_A \\ k^* &\equiv k + \delta k \\ \omega^* &\equiv \omega \\ \varphi^* &\equiv \varphi + \delta\varphi, \end{aligned} \tag{32}$$

[6] As mentioned in the footnotes of Appendix D, it is preferable to transfer the term kv_z as a shift in the z coordinate of the wave packet. See reference [8]. In this case $\delta\vec{k} = (\Delta - \delta)\,\hat{x}/v_{0x}$.

[7] Here and in Appendices B and C, instead of the usual vector notation $\vec{q}$, we use the simplified notation q, which is the matrix of the components of the vector in a given coordinate system $q = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and the notation $\tilde{k}$ which stands for the transposed matrix (k_x, k_y, k_z). So that, the scalar product $\vec{k}\cdot\vec{q}$ is written $\tilde{k}q$. The same notation is used for tensors.

where q_{cl} is the central position of the incoming atomic wave packet, where δk is the additional momentum transferred to the excited atoms out of resonance, and where $\delta\varphi$ is the phase $\delta\varphi \equiv -\delta\tilde{k}q_A$.

Let us emphasize that, in this calculation, we have never assumed that the splitter was infinitely thin or that the atom trajectory was classical.

4. CONCLUSIONS AND PERSPECTIVES: MULTIDIMENSIONAL ATOM INTERFEROMETERS

We have derived simple phase factors for the beam splitters that display explicitly the difference between the spatial and temporal cases. These phase factors have to be combined with the phase factors coming from the action integral and from the end-points splitting as discussed in Appendix C for any given interferometer geometry. This procedure has been applied in previous publications to the cases of gravimeters [7], gyros and atomic clocks [8, 10].

We have kept the calculations as simple as possible by assuming weak e.m. interactions and free-motion in the beam splitters:

$$A(t', t_1) = 1, \qquad B(t', t_1) = t' - t_1, \qquad \vec{\xi}(t', t_1) = 0. \tag{33}$$

It is clear that in realistic calculations these two assumptions have to be abandoned at the expense of more cumbersome expressions. Strong fields lead to the Borrmann effect and new corrections to the phase shifts induced by other fields have to be introduced.

In some atomic clocks, the atoms (or ions) are confined to a small region in space by an external e.m. trapping potential. This leads to a suppression of the first-order Doppler shift and of the recoil shift known as the Lamb-Dicke or Mössbauer effect. In our approach it is easy to recover such effects by the inclusion of the relevant A and B matrices in Eq. (83). If ω_T is the trap frequency

$$A(t', t_1) = \cos[\omega_T(t' - t_1)], \qquad B(t', t_1) = \frac{1}{\omega_T}\sin[\omega_T(t' - t_1)] \tag{34}$$

then the factor

$$e^{i\vec{k}'\cdot A(t',t_1)\cdot\vec{r}_1} e^{i\vec{k}'\cdot B(t',t_1)\cdot\vec{p}/M} e^{i\hbar\vec{k}' A\tilde{B}\vec{k}'/2M} \tag{35}$$

can be expanded in Bessel functions J_n and it is clear that the term associated with J_0 will be free of first-order and recoil shifts.

If, on the contrary, atoms are falling in a constant gravitational field $\vec{g}$, then

$$\vec{\xi}(t', t_1) = \frac{1}{2}\vec{g}(t' - t_1)^2 \tag{36}$$

and

$$\int_{-\infty}^{+\infty} dt' e^{i\vec{k}'\cdot\vec{g}(t'-t_1)^2/2+i[\omega_{ba}+\omega'+\vec{k}'\cdot\vec{p}/M+\hbar\vec{k}'^2/2M](t'-t_1)}$$

$$= \frac{\sqrt{2\pi}}{\sqrt{-i\vec{k}'\cdot\vec{g}}}\exp\left[-\frac{2i}{\vec{k}'\cdot\vec{g}}\left(\omega_{ba}+\omega'+\frac{\vec{k}'\cdot\vec{p}}{M}+\frac{\hbar\vec{k}'^2}{2M}\right)^2\right] \qquad (37)$$

replaces the $\delta(\omega_{ba}+\omega'+\vec{k}'\cdot\vec{p}/M+\hbar\vec{k}'^2/2M)$ function in Eq. (85) and is easily combined with Gaussians in k_x' or ω' to give the lineshape.

The previous calculations also assume that the beam splitters consist only of one laser beam in a specific privileged direction $\hat{z}$. We may extend this concept to a 2 or 3D atom-wave splitter comprising two or three laser beams in different directions (orthogonal or not). From the results of this paper, we may infer that the set of two or three beam splitters will generate clouds propagating in orthogonal directions, which have a well-defined phase relationship imposed by the orthogonal laser beams (that may come from a single laser source). The diffracted atom wave will then consist of a coherent superposition of excited state amplitudes e.g.: $b_x(\vec{r},t), b_y(\vec{r},t), b_z(\vec{r},t)$ which differ by their additional momentum $\hbar k\hat{x}, \hbar k\hat{y}, \hbar k\hat{z}$. After some time the two or three excited state wave-packets can be deflected and later recombined thus forming a multi-arms multi-dimensional interferometer. For example, if the atom wave packet travels with some initial velocity in the $\hat{x}$ direction two orthogonal laser beams in the $\hat{y}$ and $\hat{z}$ directions will generate a set of four beams $(\alpha, p_y, p_z) = (a, -\hbar k/2, -\hbar k/2)$, $(b, -\hbar k/2, \hbar k/2)$, $(b, \hbar k/2, -\hbar k/2)$, $(a, \hbar k/2, \hbar k/2)$. Two more identical beam splitters will generate a diamond-shaped interferometer. If, on the other hand, one starts with an atomic cloud at rest, three orthogonal travelling laser waves will generate a set of three diffracted clouds in the excited state

$(\alpha, p_x, p_y, p_z) = (b, \hbar k, 0, 0), (b, 0, \hbar k, 0), (b, 0, 0, \hbar k)$, thus forming an expanding inertial trihedron with the initial wave packet $(a, 0, 0, 0)$. After some time the three excited wave packets can be stopped by a second interaction while the $(a, 0, 0, 0)$ wave packet is again split into three moving pieces that will later interfere with the three previous ones. In this way a 3D version of the usual atom gravimeter can be generated. If the initial cloud is cold enough (sub-recoil) or by accumulating many recoils [20, 21], the three interfering clouds can be resolved in space and give three independent fringe patterns. Alternatively, phases, polarizations, frequencies and time delays of each one of the laser beams can be used to discriminate between the various interferometers formed by the each pair of atomic paths. One can also use counterpropagating laser beams to bring back the three diffracted clouds to the origin and generate a 3-D Bordé-Ramsey optical clock. By varying the orientations many spurious phases [22] can be cancelled.

APPENDIX A

A Relativistic Schrödinger-Type Equation for Atom Waves

Atoms in a given internal energy state can be treated as quanta of a matter-wave field with a rest mass M corresponding to this internal energy and a spin corresponding to the total angular momentum in that state. To take this spin into account one can use, for example, a Dirac [24, 25], Proca or higher-spin wave equation. Here, for simplicity, we shall ignore this spin and start simply with the Klein-Gordon equation for the covariant wave amplitude of a scalar field:

$$\left[\Box + \frac{M^2c^2}{\hbar^2}\right]\varphi = 0, \tag{38}$$

where the d'Alembertian is related to the curved space-time metric $g^{\mu\nu}$ by the usual expression

$$\Box\varphi = g^{\mu\nu}\nabla_\mu\nabla_\nu\varphi = (-g)^{-1/2}\,\partial_\mu\left[(-g)^{1/2}\,g^{\mu\nu}\partial_\nu\varphi\right]. \tag{39}$$

We assume that space-time admits a coordinate system (x^μ) in which the metric tensor takes the form

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| \ll 1. \tag{40}$$

In what follows, the $h_{\mu\nu}$'s will be considered as first-order quantities and all calculations will be valid at this order, e.g.

$$\sqrt{-g} = 1 + \frac{h}{2} \qquad \text{with} \qquad h = h^\mu{}_\mu = \eta_{\mu\nu}h^{\mu\nu}. \tag{41}$$

Then the Klein-Gordon equation becomes

$$\left[\partial^\mu\partial_\mu + \frac{M^2c^2}{\hbar^2}\right]\varphi + \frac{1}{2}(\partial_\mu h)\partial^\mu\varphi - \partial_\mu h^{\mu\nu}\partial_\nu\varphi = 0. \tag{42}$$

We shall furthermore assume that the covariant amplitude has the form

$$\varphi = \varphi_0 \exp\left[-i\frac{E_0 t}{\hbar}\right], \tag{43}$$

where φ_0 varies slowly with time. Then

$$\frac{\partial^2\varphi}{\partial t^2} \simeq -2i\frac{E_0}{\hbar}\frac{\partial\varphi}{\partial t} + \frac{E_0^2}{\hbar^2}\varphi \tag{44}$$

and one obtains a Schrödinger-like equation (after renormalization to take into account the change in scalar product)[8]:

$$i\hbar\frac{\partial\varphi}{\partial t} = \left(\frac{E_0}{2} + \frac{M^2c^4}{2E_0}\right)\varphi - \frac{\hbar^2c^2}{2E_0}\nabla^2\varphi - \frac{\hbar^2c^2}{2E_0}\partial_\mu h^{\mu\nu}\partial_\nu\varphi \tag{47}$$

or in the momentum representation

$$i\hbar\frac{\partial\varphi}{\partial t} = \left(\frac{E_0}{2} + \frac{M^2c^4}{2E_0}\right)\varphi - \frac{c^2}{2E_0}p^j p_j\varphi + \frac{c^2}{2E_0}p_\mu h^{\mu\nu}p_\nu\varphi \tag{48}$$

This means that the usual hyperbolic dispersion curve is locally approximated by the parabola tangent to the hyperbola for the energy E_0. This approximation scheme applies to massive as well as to massless particles (e.g. for quasi-monochromatic light $M = 0$ and $E_0 = \hbar\omega$ [19]). However, in this limit, only the group velocity of a wave packet is correct, wheras the longitudinal wave-packet spreading requires higher-order terms (p^4) in the expansion of $\sqrt{1 + (p^2 - p_0^2)c^2/E_0^2}$. This slowly varying phase and amplitude approximation can even be used when the weak-field approximation is not valid. To first-order, the Linet-Tourrenc phase shift [26] is immediately recovered. If we introduce the mass M^* defined by:

$$E_0 = M^*c^2 \tag{49}$$

the field equation can be written as an ordinary Schrödinger equation in flat space-time

$$i\hbar\frac{\partial\varphi}{\partial t} = \frac{M^*c^2}{2}\left(1 + \frac{M^2}{M^{*2}}\right)\varphi - \frac{1}{2M^*}p^j p_j\varphi + \frac{1}{2M^*}p_\mu h^{\mu\nu}p_\nu\varphi. \tag{50}$$

The non-relativistic limit is obtained for $M^* \to M$. This equation can also be written as

$$\begin{aligned} i\hbar\frac{\partial\varphi}{\partial t} - \frac{1}{2}\left(E_0h^{00} + \frac{c}{2}\left(p_ih^{i0} + h^{i0}p_i\right)\right)\varphi &= \left(\frac{E_0}{2} + \frac{M^2c^4}{2E}\right)\varphi \\ -\frac{1}{2E_0}\left(cp^j - \frac{1}{2}\left(E_0h^{j0} + cp_ih^{ij}\right)\right)\left(cp_j - \frac{1}{2}\left(E_0h_{j0} + ch_{ij}p^i\right)\right)\varphi & \end{aligned} \tag{51}$$

to display the analogs of the scalar and vector e.m. potentials as in [25].

[8] The rule

$$\partial_t \to -iE_0/\hbar \tag{45}$$

$$p_0 = E/c \to E_0/c \tag{46}$$

is used in the terms associated with $h^{\mu\nu}$.

Eq.(42) is invariant under the infinitesimal coordinate transformation (gauge invariance)

$$x^{\mu} \to x^{\mu} + \xi^{\mu} \tag{52}$$

i.e. under the simultaneous changes of $\varphi \to [1 - \xi^{\mu}\partial_{\mu}]\varphi$ and $h_{\mu\nu} \to h_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu}$. The corresponding finite gauge transformation

$$\mathcal{T} \exp\left[\frac{i}{\hbar}\int p^{\mu} X_{\mu\nu} dx^{\nu}\right]\varphi, \tag{53}$$

where $\mathcal{T}$ is an ordering operator and where the quantities $X_{\mu\nu}$ are gauge functions, suggests the general transformation

$$U = \exp\left[\frac{i}{\hbar}\Lambda(t)\right] \mathcal{T} \exp\left[\frac{i}{\hbar E_0}\int p^{\mu} X_{\mu\nu}(x) p^{\nu} dt\right] \tag{54}$$

in order to remove the gravito-inertial interaction terms in Schrödinger equation. This is, indeed, what is performed in references [8, 19].

APPENDIX B

Background on the ABCD Matrices[9]

In most cases of interest for atom interferometry, the external motion Hamiltonian (i.e. relative to the center-of-mass motion) can be expressed as a quadratic polynomial of momentum and position operators

$$H_{\text{ext}} = \frac{1}{2}\vec{p}_{\text{op}} \cdot \overset{\Rightarrow}{\alpha}(t) \cdot \vec{q}_{\text{op}} + \frac{1}{2M^*}\vec{p}_{\text{op}} \cdot \overset{\Rightarrow}{\beta}(t) \cdot \vec{p}_{\text{op}} - \frac{1}{2}\vec{q}_{\text{op}} \cdot \overset{\Rightarrow}{\delta}(t) \cdot \vec{p}_{\text{op}}$$
$$- \frac{M^*}{2}\vec{q}_{\text{op}} \cdot \overset{\Rightarrow}{\gamma}(t) \cdot \vec{q}_{\text{op}} + \vec{f}(t) \cdot \vec{p}_{\text{op}} - M^*\vec{g}(t) \cdot \vec{q}_{\text{op}}. \tag{55}$$

The evolution of wave packets under the influence of this Hamiltonian has been studied in detail and is given by the $ABCD$ law. But, we know from Ehrenfest theorem, that the motion of a wave packet is also obtained in this case from classical equations. The equations satisfied by the $ABCD$ matrices can be derived either from the Hamilton-Jacobi equation (see [7]) or from Hamilton's equations. For the previous Hamiltonian, Hamilton's equations can be written as an equation for the two-component vector

$$\chi = \begin{pmatrix} q \\ p/M^* \end{pmatrix} \tag{56}$$

as

$$\frac{d\chi}{dt} = \begin{pmatrix} \frac{dH_{\text{ext}}}{dp} \\ -\frac{1}{M^*}\frac{dH_{\text{ext}}}{dq} \end{pmatrix} = \Gamma(t)\chi + \Phi(t), \tag{57}$$

[9] Based on [7, 8, 19].

where

$$\Gamma(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \gamma(t) & \delta(t) \end{pmatrix} \tag{58}$$

is a time-dependent 6×6 matrix, with (Hermiticity of the Hamiltonian)

$$\delta(t) = -\tilde{\alpha}(t) \tag{59}$$

(for a pure rotation we have $\alpha(t) = \delta(t) = i\vec{J} \cdot \vec{\Omega}$), and where

$$\Phi(t) = \begin{pmatrix} f(t) \\ g(t) \end{pmatrix}. \tag{60}$$

The integral of Hamilton's equation can thus be written as

$$\chi(t) = \begin{pmatrix} A(t,t_0) & B(t,t_0) \\ C(t,t_0) & D(t,t_0) \end{pmatrix} \chi(t_0) + \begin{pmatrix} \xi(t,t_0) \\ \phi(t,t_0) \end{pmatrix}, \tag{61}$$

where

$$\mathcal{M}(t,t_0) = \begin{pmatrix} A\,(t,t_0) & B\,(t,t_0) \\ C\,(t,t_0) & D\,(t,t_0) \end{pmatrix} = \mathcal{T} \exp\left[\int_{t_0}^{t} \begin{pmatrix} \alpha(t') & \beta(t') \\ \gamma(t') & \delta(t') \end{pmatrix} dt'\right], \tag{62}$$

with $\mathcal{T}$ as time-ordering operator, and where

$$\begin{pmatrix} \xi(t,t_0) \\ \phi(t,t_0) \end{pmatrix} = \int_{t_0}^{t} \mathcal{M}(t,t')\Phi(t')dt'. \tag{63}$$

One can easily show that

$$\phi = \beta^{-1}(\dot{\xi} - \alpha\xi - f). \tag{64}$$

As an illustration, one can calculate the $ABCD$ matrix in the case of gravitational waves:

- in Einstein coordinates:

$$\overset{\Rightarrow}{\beta}(t) = \overset{\Rightarrow}{h} \cos(\omega_{\rm gw} t + \varphi), \qquad \overset{\Rightarrow}{\gamma}(t) = 0, \tag{65}$$

  where $\overset{\Rightarrow}{h} = \{h^{ij}\}$ and where $\omega_{\rm gw}$ is a gravitational wave frequency.
- in Fermi coordinates:

$$\overset{\Rightarrow}{\beta}(t) = \overset{\Rightarrow}{1}, \qquad \overset{\Rightarrow}{\gamma}(t) = \left(\omega_{\rm gw}^2/2\right) \overset{\Rightarrow}{h} \cos(\omega_{\rm gw} t + \varphi), \tag{66}$$

  where the z dependence of the wave is contained in φ.

Then, from the formulas given above, to first-order in h:

- in Einstein coordinates:

$$A = 1, \qquad B = t + \frac{h}{\omega_{\mathrm{gw}}}[\sin(\omega_{\mathrm{gw}} t + \varphi) - \sin\varphi] \tag{67}$$

- in Fermi coordinates:

$$A = 1 - \frac{h}{2}[\cos(\omega_{\mathrm{gw}} t + \varphi) - \cos\varphi] - \frac{h\omega_{\mathrm{gw}} t}{2}\sin\varphi, \tag{68}$$

$$B = t + \frac{h}{\omega_{\mathrm{gw}}}[\sin(\omega_{\mathrm{gw}} t + \varphi) - \sin\varphi] - \frac{ht}{2}[\cos(\omega_{\mathrm{gw}} t + \varphi) + \cos\varphi]. \tag{69}$$

APPENDIX C

Phase-Shift Formula for Atom Interferometers

The total phase difference between both arms of an interferometer is the sum of three terms: the difference in the action integral along each path, the difference in the phases imprinted on the atom waves by the beam splitters and a contribution coming from the splitting of the wave packets at the exit of the interferometer [7]. If α and β are the two branches of the interferometer

$$\begin{aligned}\delta\phi(q) = {} & \frac{1}{\hbar}\sum_{j=1}^{N}[S_\beta(t_{j+1}, t_j) - S_\alpha(t_{j+1}, t_j)] \\ & + \sum_{j=1}^{N}(\tilde{k}_{\beta j} q_{\beta j} - \tilde{k}_{\alpha j} q_{\alpha j}) - (\omega_{\beta j} - \omega_{\alpha j})t_j + (\varphi_{\beta j} - \varphi_{\alpha j}) \\ & + \frac{1}{\hbar}[\tilde{p}_{\beta,D}(q - q_{\beta,D}) - \tilde{p}_{\alpha,D}(q - q_{\alpha,D})]\end{aligned} \tag{70}$$

where $S_{\alpha j} = S_\alpha(t_{j+1}, t_j)$ and $S_{\beta j} = S_\beta(t_{j+1}, t_j)$.

In the case of quadratic Hamiltonians, the four end-points theorem derived in [9] states that along homologous segments of the two branches (where τ_j is a proper time)

$$\begin{aligned}& \frac{S_{\alpha j}}{M_{\alpha j}} + \frac{\tilde{p}_{\alpha,j+1}}{2M_{\alpha j}}(q_{\beta,j+1} - q_{\alpha,j+1}) - \frac{\tilde{p}_{\alpha j} + \hbar\tilde{k}_{\alpha j}}{2M_{\alpha j}}(q_{\beta j} - q_{\alpha j}) \\ & = \frac{S_{\beta j}}{M_{\beta j}} + \frac{\tilde{p}_{\beta,j+1}}{2M_{\beta j}}(q_{\alpha,j+1} - q_{\beta,j+1}) - \frac{\tilde{p}_{\beta j} + \hbar\tilde{k}_{\beta j}}{2M_{\beta j}}(q_{\alpha j} - q_{\beta j}) \\ & = -c^2\tau_j\end{aligned} \tag{71}$$

from which we get

$$S_{\beta j} - S_{\alpha j} = \frac{1}{2}(\tilde{p}_{\beta,j+1} + \tilde{p}_{\alpha,j+1})(q_{\beta,j+1} - q_{\alpha,j+1})$$
$$- \frac{1}{2}(\tilde{p}_{\beta,j} + \tilde{p}_{\alpha,j})(q_{\beta,j} - q_{\alpha,j}) - \frac{\hbar}{2}(\tilde{k}_{\beta,j} + \tilde{k}_{\alpha,j})(q_{\beta,j} - q_{\alpha,j})$$
$$- (M_{\beta j} - M_{\alpha j})c^2\tau_j \tag{72}$$

and

$$\delta\phi(q) = \sum_{j=1}^{N}(\tilde{k}_{\beta j}q_{\beta j} - \tilde{k}_{\alpha j}q_{\alpha j}) - \frac{1}{2}(\tilde{k}_{\beta j} + \tilde{k}_{\alpha j})(q_{\beta j} - q_{\alpha j})$$
$$+ \sum_{j=1}^{N}\left[\omega_{\beta\alpha j}(t_{j+1} - t_j) - \omega_{\beta\alpha j}^{(0)}\tau_j\right] + \sum_{j=1}^{N}(\varphi_{\beta j} - \varphi_{\alpha j})$$
$$+ \frac{(\tilde{p}_{\beta D} - \tilde{p}_{\alpha D})}{\hbar}\left(q - \frac{q_{\beta D} + q_{\alpha D}}{2}\right) - \frac{\tilde{p}_{\alpha 1} + \tilde{p}_{\beta 1}}{2\hbar}(q_{\beta 1} - q_{\alpha 1}) \tag{73}$$

with $\omega_{\beta\alpha j} = \sum_{k=1}^{j}\omega_{\beta k} - \omega_{\alpha k}$ and $\omega_{\beta\alpha j}^{(0)} = (M_{\beta j} - M_{\alpha j})c^2/\hbar$.

Usually $q_{\beta 1} = q_{\alpha 1}$ and we may use the mid-point theorem [8] which states that the phase difference for the fringe signal integrated over space is given by the phase difference before integration at the mid-point $(q_{\beta,D} + q_{\alpha,D})/2$, so that the last line of the previous equation drops out. In the case of identical masses, we see that the contributions of the action and of the end points splitting (except for small recoil corrections proportional to k^2) have cancelled each other and we are left with the contributions from the beam splitters only.

For a symmetric Bordé interferometer (Mach-Zehnder diamond geometry) $k_{\beta i} + k_{\alpha i} = 0,\ \forall i \in [2, N-1]$, and with the approximation of equal masses $M_{\beta i} = M_{\alpha i} = M$ the following simple result is obtained

$$\delta\phi = \sum_{j=1}^{N}\Big[(\tilde{k}_{\beta j}q_{\beta j} - \tilde{k}_{\alpha j}q_{\alpha j}) + (\tilde{k}_{\beta N} + \tilde{k}_{\alpha N})\frac{q_{\alpha N} - q_{\beta N}}{2}$$
$$- (\omega_{\beta j} - \omega_{\alpha j})t_j + (\varphi_{\beta j} - \varphi_{\alpha j})\Big]$$
$$= \sum_{j=1}^{N}\left[(\tilde{k}_{\beta j} - \tilde{k}_{\alpha j})\frac{q_{\alpha j} + q_{\beta j}}{2} - (\omega_{\beta j} - \omega_{\alpha j})t_j + (\varphi_{\beta j} - \varphi_{\alpha j})\right] \tag{74}$$

which is manifestly gauge-invariant. The coordinates $q_{\alpha j}$ and $q_{\beta j}$ are finally calculated with the $ABCD$ matrices. As an example, in the case of three beam splitters only

$$\delta\phi = [\tilde{k}_1 - 2\tilde{k}_2 A(t_2, t_1) + \tilde{k}_3 A(t_3, t_1)] q_1 \\ + [\tilde{k}_3 B(t_3, t_1) - 2\tilde{k}_2 B(t_2, t_1)] \left(\frac{p_1}{M} + \frac{\hbar k_1}{2M}\right) + \varphi_1 - 2\varphi_2 + \varphi_3 \tag{75}$$

which, for equal time intervals T, frequencies and wave vectors k, gives

$$\delta\phi = \tilde{k}\,[1 - 2A\,(T) + A\,(2T)]\,q_1 \\ + \tilde{k}\,[B\,(2T) - 2B\,(T)] \left(\frac{p_1}{M} + \frac{\hbar k}{2M}\right) + \varphi_1 - 2\varphi_2 + \varphi_3. \tag{76}$$

As an illustration, one can calculate this phase shift in the case of gravitational waves

$$\begin{aligned}\delta\phi &= -\frac{\tilde{k}hq_1}{2}[\cos(2\omega_{\mathrm{gw}}T + \varphi) - 2\cos(\omega_{\mathrm{gw}}T + \varphi) + \cos\varphi] \\ &\quad + \frac{\tilde{k}h}{\omega_{\mathrm{gw}}} V_1 [\sin(2\omega_{\mathrm{gw}}T + \varphi) - 2\sin(\omega_{\mathrm{gw}}T + \varphi) + \sin\varphi] \\ &\quad - \tilde{k}hV_1 T[\cos(2\omega_{\mathrm{gw}}T + \varphi) - \cos(\omega_{\mathrm{gw}}T + \varphi)] + \varphi_1 - 2\varphi_2 + \varphi_3 \\ &= \tilde{k}\gamma q_1 T^2 \frac{\sin^2(\omega_{\mathrm{gw}}T/2)}{(\omega_{\mathrm{gw}}T/2)^2} - \tilde{k}hV_1\omega_{\mathrm{gw}}T^2 \sin(\omega_{\mathrm{gw}}T + \varphi)\frac{\sin^2(\omega_{\mathrm{gw}}T/2)}{(\omega_{\mathrm{gw}}T/2)^2} \\ &\quad - \tilde{k}hV_1 T[\cos(2\omega_{\mathrm{gw}}T + \varphi) - \cos(\omega_{\mathrm{gw}}T + \varphi)] + \varphi_1 - 2\varphi_2 + \varphi_3,\end{aligned} \tag{77}$$

where

$$V_1 = \frac{1}{M}\left(p_1 + \frac{\hbar k}{2}\right) \qquad \text{and} \qquad \gamma = \frac{\omega_{\mathrm{gw}}^2}{2} h\cos(\omega_{\mathrm{gw}}T + \varphi). \tag{78}$$

The first term is the phase shift already derived in [28]. It corresponds to the action of the gravitational wave on the light beam connecting the two atomic clouds in a gradiometer set-up. The formula satisfies the equivalence principle. It reduces to that derived for the atom gravimeter in [7] in the static limit and is very similar to the formula derived for the Sagnac effect in [8].

APPENDIX D

First-Order Excited State Amplitude for One-Photon Transitions

In the position representation, the first-order excited state amplitude

$$\left\langle b|\Psi^{(1)}(t)\right\rangle = \frac{1}{i\hbar}\langle b|U_0(t, t_1)|b\rangle \int_{t_o}^{t} dt' V_{ba}^{+-}(\vec{R}_{\mathrm{op}}(t', t_1), t')e^{i\omega_{ba}(t'-t_1)} \\ \langle a|U_0(t_1, t_0)|a\rangle\left\langle a|\Psi^{(0)}(t_0)\right\rangle \tag{79}$$

gives the following amplitude for the scattered wave packet

$$\begin{aligned} b^{(1)}(\vec{r},t) &= \langle \vec{r}, b|\Psi^{(1)}(t)\rangle \\ &= \int d^3r_1 \langle \vec{r}, b|U_0(t,t_1)|\vec{r}_1, b\rangle \frac{1}{i\hbar}\int_{-\infty}^{+\infty} dt' e^{i\omega_{ba}(t'-t_1)} \\ &\quad \int d^3p \langle \vec{r}_1|V_{ba}^{+-}(\vec{R}_{\text{op}}(t',t_1),t')|\vec{p}\rangle\langle \vec{p}, a|U_0(t_1,t_0)|\Psi^{(0)}(t_0)\rangle, \end{aligned} \tag{80}$$

where we have let t and t_0 go to infinity[10] (bounded interaction in space or time). Let us introduce, as an intermediate step

$$\begin{aligned} b_{\text{eff}}^{(1)}(\vec{r}_1,t_1) &= \frac{1}{i\hbar}\int_{-\infty}^{+\infty} dt' e^{i\omega_{ba}(t'-t_1)} \\ &\quad \int d^3p \langle \vec{r}_1 \big| V_{ba}^{+-}(\vec{R}_{\text{op}}(t',t_1),t') \big| \vec{p}\rangle\langle \vec{p}, a|U_0(t_1,t_0)|\Psi^{(0)}(t_0)\rangle \\ &= \frac{1}{i\hbar}\int_{-\infty}^{+\infty} dt' e^{i\omega_{ba}(t'-t_1)} \int d^3p \int \frac{d^3k'd\omega'}{(2\pi)^2} V_{ba}^{+-}(\vec{k}',\omega') \\ &\quad \langle \vec{r}_1 \big| e^{i\vec{k}'\cdot(A(t',t_1)\cdot\vec{r}_{\text{op}}+B(t',t_1)\cdot\vec{p}_{\text{op}}/M+\vec{\xi}(t',t_1))+i\omega't'} \big| \vec{p}\rangle \\ &\quad \langle \vec{p}, a|U_0(t_1,t_0)|\Psi^{(0)}(t_0)\rangle. \end{aligned} \tag{81}$$

This effective scattered amplitude term will be later propagated in the absence of V from $(\vec{r}_1, t_1)$ to $(\vec{r}, t)$

$$b^{(1)}\left(\vec{r},t\right) = \int d^3r_1 \langle \vec{r}, b|U_0(t,t_1)|\vec{r}_1, b\rangle b_{\text{eff}}^{(1)}(\vec{r}_1,t_1). \tag{82}$$

We check that this final amplitude is indeed independent of t_1 in the case of free propagation (we shall drop the subscript "eff" in what follows)

$$\begin{aligned} b^{(1)}(\vec{r}_1,t_1) &= \frac{1}{i\hbar}\int_{-\infty}^{+\infty} dt' e^{i\omega_{ba}(t'-t_1)} \int d^3p \int \frac{d^3k'd\omega'}{(2\pi)^2} V_{ba}^{+-}(\vec{k}',\omega') \\ &\quad e^{i\vec{k}'\cdot(A(t',t_1)\cdot\vec{r}_1+\vec{\xi}(t',t_1))+i\omega't'} e^{i\vec{k}'\cdot B(t',t_1)\cdot\vec{p}/M} e^{i\hbar\vec{k}'A\tilde{B}\vec{k}'/2M} \\ &\quad \langle \vec{r}_1|\vec{p}\rangle\langle \vec{p}, a|U_0(t_1,t_0)|\Psi^{(0)}(t_0)\rangle \\ &= \frac{1}{i\hbar}\int \frac{d^3k'd\omega'}{(2\pi)^2} V_{ba}^{+-}(\vec{k}',\omega') e^{-i\omega't_1} \int d^3p \\ &\quad \int_{-\infty}^{+\infty} dt' e^{[i(\omega_{ba}+\omega')(t'-t_1)+i\vec{k}'\cdot B(t',t_1)\cdot\vec{p}/M+i\vec{k}'\cdot\vec{\xi}(t',t_1)+i\hbar\vec{k}'A\tilde{B}\vec{k}'/2M]} \\ &\quad e^{i\vec{k}'\cdot A(t',t_1)\cdot\vec{r}_1} \langle \vec{r}_1|\vec{p}\rangle\langle \vec{p}, a|U_0(t_1,t_0)|\Psi^{(0)}(t_0)\rangle. \end{aligned} \tag{83}$$

[10] Calculations could also be pursued with a time integral from $-\infty$ to t as in references [7, 8, 13], see the calculation in the two-photon case in Appendix E.

Let us assume free-motion in the beam-splitter

$$A(t', t_1) = 1, \qquad B(t', t_1) = t' - t_1, \qquad \vec{\xi}(t', t_1) = 0. \tag{84}$$

Then

$$\begin{aligned}
b^{(1)}(\vec{r}_1, t_1) &= \frac{1}{i\hbar}\int_{-\infty}^{+\infty} dt' e^{i\omega_{ba}(t'-t_1)} \\
&\quad \int d^3p \langle \vec{r}_1 | V_{ba}^{+-}(\vec{R}_{\mathrm{op}}(t', t_1), t') | \vec{p} \rangle \langle \vec{p}, a | U_0(t_1, t_0) | \Psi^{(0)}(t_0) \rangle \\
&= \frac{1}{2\pi i\hbar}\int d^3k' d\omega' V_{ba}^{+-}(\vec{k}', \omega') e^{i\vec{k}'\cdot\vec{r}_1 + i\omega' t_1} \int d^3p \\
&\quad \delta\left(\omega_{ba} + \omega' + \vec{k}'\cdot\vec{p}/M + \hbar\vec{k}'^2/2M\right) \\
&\quad \langle \vec{r}_1 | \vec{p} \rangle \langle \vec{p}, a | U_0(t_1, t_0) | \Psi^{(0)}(t_0) \rangle.
\end{aligned} \tag{85}$$

With the expression Eq. (20) for $V_{ba}^{+-}(\vec{k}', \omega')$

$$\begin{aligned}
b^{(1)}(\vec{r}_1, t_1) &= i\Omega_{ba}^{+} e^{-i(\omega t_A + \varphi^+)} \frac{w_0^2}{2}\int d^3p \int \frac{d\omega'}{\sqrt{2\pi}} \tilde{F}(\omega' + \omega) e^{i\omega'(t_1 - t_A)} \\
&\quad \int d^2k'_\perp \tilde{U}_0(\vec{k}'_\perp) e^{i\vec{k}'_\perp\cdot(\vec{r}_1 - \vec{r}_A)} e^{ikz_1 - ik_\perp'^2(z_1 - z_A)/2k} \\
&\quad \delta\left(\omega_{ba} + \omega' + \left(k - k_\perp'^2/2k\right)p_z/M + \vec{k}'_\perp\cdot\vec{p}/M + \delta\right) \\
&\quad \langle \vec{r}_1 | \vec{p} \rangle \langle \vec{p}, a | U_0(t_1, t_0) | \Psi^{(0)}(t_0) \rangle,
\end{aligned} \tag{86}$$

where the recoil term $\hbar\vec{k}'^2/2M$ is approximated by $\delta = \hbar k^2/2M$. Next we perform the ω' integration

$$\begin{aligned}
b^{(1)}(\vec{r}_1, t_1) &= i\Omega_{ba}^{+} e^{-i(\omega t_A + \varphi^+)} \frac{w_0^2}{2\sqrt{2\pi}} \int d^3p \int d^2k'_\perp \\
&\quad \tilde{F}\left(\omega - \omega_{ba} - \left(k - k_\perp'^2/2k\right)p_z/M - \vec{k}'_\perp\cdot\vec{p}/M - \delta\right) \\
&\quad e^{-i[\omega_{ba} + (k - k_\perp'^2/2k)p_z/M + \vec{k}'_\perp\cdot\vec{p}/M + \delta](t_1 - t_A)} \\
&\quad \tilde{U}_0^{+}(\vec{k}'_\perp) e^{i\vec{k}'_\perp\cdot(\vec{r}_1 - \vec{r}_A)} e^{ikz_1 - ik_\perp'^2(z_1 - z_A)/2k} \\
&\quad \langle \vec{r}_1 | \vec{p} \rangle \langle \vec{p}, a | U_0(t_1, t_0) | \Psi^{(0)}(t_0) \rangle.
\end{aligned} \tag{87}$$

This result may be simplified with the choice $t_1 = t_A$. If we neglect also the dispersive character coming from the momentum dependence in the envelope $\tilde{F}$, then the following simple result is obtained

$$b^{(1)}(\vec{r}_1, t_1) = i\Omega_{ba}^{+} e^{-i(\omega t_A+\varphi^+)} e^{ikz_1} \frac{w_0^2}{2\sqrt{2\pi}} \int d^2k'_\perp$$
$$\tilde{F}\left(\omega - \omega_{ba} - \left(k - k'^2_\perp/2k\right) v_{0z} - \vec{k'_\perp} \cdot \vec{v}_0 - \delta\right)$$
$$\tilde{U}_0^{+}(\vec{k'_\perp}) e^{-ik'^2_\perp (z_1 - z_A)/2k} e^{i\vec{k'_\perp} \cdot (\vec{r}_1 - \vec{r_A})} \left\langle \vec{r}_1, a | \Psi^{(0)}(t_1) \right\rangle. \qquad (88)$$

However, we shall postpone these two choices and first show how the $k'_\perp$ integration can be performed. To simplify, we keep only the k'_x term (assuming $p_y = 0$) and neglect the quadratic correction to k in $\tilde{F}$

$$b^{(1)}(\vec{r}_1, t_1) = i\Omega_{ba}^{+} \frac{w_0}{\sqrt{2}} e^{-i(\omega t_A+\varphi^+)} e^{ikz_1} G^{+*}(y_1 - y_A, z_1 - z_A)$$
$$\int d^3 p e^{-i[\omega_{ba}+kp_z/M+\delta](t_1-t_A)} \int dk'_x$$
$$\tilde{F}(\omega - \omega_{ba} - kp_z/M - k'_x p_x/M - \delta)$$
$$\tilde{G}_0(k'_x) e^{-ik'^2_x (z_1 - z_A - p_z/M(t_1-t_A))/2k} e^{ik'_x (x_1 - x_A - p_x/M(t_1-t_A))}$$
$$\langle \vec{r}_1 | \vec{p} \rangle \left\langle \vec{p}, a | U_0(t_1, t_0) | \Psi^{(0)}(t_0) \right\rangle, \qquad (89)$$

where we have introduced factorized x and y dependences

$$G^{+}(x - x_A, z - z_A) = \sqrt{L^{+}(z - z_A)} \exp\left[-L^{+}(z - z_A)(x - x_A)^2 / w_0^2\right]$$
$$= \frac{w_0}{2\sqrt{\pi}} \int dk_x \tilde{G}_0(k_x) e^{ik_x^2 (z - z_A)/2k} e^{ik_x(x - x_A)}$$
$$= \frac{w_0}{2\sqrt{\pi}} \int dk_x \tilde{G}^{+}(k_x) e^{ik_x(x - x_A)}, \qquad (90)$$

whhich is consistent with

$$U_0^{\pm}(\vec{r}) = G^{\pm}(x, z)\, G^{\pm}(y, z) \qquad (91)$$
$$\tilde{U}_0(\vec{k'_\perp}) = \tilde{G}_0(k_x) \tilde{G}_0(k_y) \quad \text{real.} \qquad (92)$$

In order to evaluate (89) we will use the convolution theorem

$$\frac{w_0}{\sqrt{2}} \int dk'_x \tilde{F}(\omega - \omega_{ba} - kv_z - k'_x v_x - \delta)$$
$$\tilde{G}_0(k'_x) e^{-i\frac{k'^2_x}{2k}(z_1 - z_A - v_z(t_1 - t_A))} e^{ik'_x(x_1 - x_A - v_x(t_1-t_A))} =$$
$$\int_{-\infty}^{+\infty} d\theta e^{i(\omega - \omega_{ba} - kv_z - \delta)\theta} F(\theta)$$
$$G^{+*}(x_1 - x_A - v_x(t_1 - t_A) - v_x\theta, z_1 - z_A - v_z(t_1 - t_A)). \qquad (93)$$

To proceed with a concrete example, we assume that the temporal envelope is a rectangular pulse (this is a frequent choice in actual experiments; another realistic choice is a pulse with a Gaussian shape)

$$F(t - t_A) = \Upsilon\left(t - t_A + \frac{\tau}{2}\right) - \Upsilon\left(t - t_A - \frac{\tau}{2}\right), \tag{94}$$

where Υ is Heaviside step function and

$$\tilde{F}(\omega' + \omega) = \sqrt{\frac{2}{\pi}} \frac{\sin[(\omega' + \omega)\tau/2]}{\omega' + \omega}, \tag{95}$$

then

$$\begin{aligned}
&\int_{-\infty}^{+\infty} d\theta e^{i(\omega-\omega_{ba}-kv_z-\delta)\theta} F(\theta) \\
&G^{+*}\left((x_1 - x_A) - v_x(t_1 - t_A) - v_x\theta, z_1 - z_A - v_z(t_1 - t_A)\right) \\
&= \int_{-\tau/2}^{+\tau/2} d\theta e^{i(\Delta-kv_z-\delta)\theta} \\
&\sqrt{L^{+*}} \exp\left[-L^{+*}\left((x_1 - x_A) - v_x(t_1 - t_A) - v_x\theta\right)^2 / w_0^2\right] \\
&= \frac{\sqrt{\pi} w_0}{2v_x} e^{ik_x(x_1 - x_A - v_x(t_1 - t_A))} \exp\left[-\frac{(k_x w_0)^2}{4L^{+*}}\right] [\mathrm{erf}(\mathcal{L}_+) - \mathrm{erf}(\mathcal{L}_-)]
\end{aligned} \tag{96}$$

with

$$k_x = (\Delta - kv_z - \delta)/v_x \tag{97}$$

and the abbreviation

$$\mathcal{L}_\pm = \sqrt{L^{+*}} \frac{x_1 - x_A - v_x(t_1 - t_A) \pm \frac{1}{2} v_x \tau}{w_0} + i \frac{(\Delta - kv_z - \delta) w_0}{2\sqrt{L^{+*}} v_x}. \tag{98}$$

Spatial Beam Splitter. For $\tau \longrightarrow +\infty$ the θ integral yields

$$\frac{\sqrt{\pi} w_0}{v_x} e^{ik_x(x_1 - x_A - v_x(t_1 - t_A))} \exp\left[-\frac{(k_x w_0)^2}{4L^{+*}}\right] \tag{99}$$

and we obtain for the continuous spatial beam splitter

$$\begin{aligned}
b^{(1)}(\vec{r}_1, t_1) = {}& i\Omega_{ba}^+ e^{-i(\omega t_A + \varphi^+)} e^{ikz_1} G^{+*}(y_1 - y_A, z_1 - z_A) \\
& \int d^3p e^{-i[\omega_{ba} + kp_z/M + \delta](t_1 - t_A)}
\end{aligned}$$

$$\frac{\sqrt{\pi}w_0}{v_x}\exp\left[ik_x\left(x_1-x_A-v_x\left(t_1-t_A\right)\right)\right]\exp\left[-\frac{\left(k_x w_0\right)^2}{4L^{+*}}\right]$$
$$\langle\vec{r}_1|\vec{p}\rangle\left\langle\vec{p},a|U_0(t_1,t_0)|\Psi^{(0)}(t_0)\right\rangle. \tag{100}$$

We check that t_A disappears

$$b^{(1)}(\vec{r}_1,t_1)=i\Omega_{ba}^{+}e^{-i(\omega t_1+\varphi^+)}e^{ikz_1}G^{+*}\left(y_1-y_A,z_1-z_A\right)$$
$$\int d^3p\frac{\sqrt{\pi}w_0}{v_x}e^{ik_x(x_1-x_A)}\exp\left[-\frac{\left(k_x w_0\right)^2}{4L^{+*}}\right]$$
$$\langle\vec{r}_1|\vec{p}\rangle\left\langle\vec{p},a|U_0(t_1,t_0)|\Psi^{(0)}(t_0)\right\rangle \tag{101}$$

and if we neglect the dispersion of the splitter:

$$b^{(1)}(\vec{r}_1,t_1)=i\sqrt{\pi}\left(\frac{\Omega_{ba}^{+}w_0}{v_{0x}}\right)e^{-i(\omega t_1-kz_1+\varphi^+)}G^{+*}\left(y_1-y_A,z_1-z_A\right)$$
$$e^{ik_{0x}(x_1-x_A)}\exp\left[-\frac{\left(k_{0x}w_0\right)^2}{4L^{+*}}\right]a^{(0)}(\vec{r}_1,t_1) \tag{102}$$

where

$$a^{(0)}(\vec{r},t)=\left\langle\vec{r},a|\Psi^{(0)}(t)\right\rangle \tag{103}$$

is the unperturbed (that is, for the absence of the e.m. field) ground-state wave packet amplitude, and where

$$k_{0x}=\frac{\Delta-kv_{0z}-\delta}{v_{0x}} \tag{104}$$

is the momentum communicated to the atom out of resonance. Here v_{0x} and v_{0z} are the velocity components of the wave packet center[11].

Temporal Beam Splitter. If v_x and $v_z\longrightarrow 0$ (or $w_0\longrightarrow+\infty$), then the θ integral gives

$$\frac{\sin\left((\Delta-kv_z-\delta)\tau/2\right)}{(\Delta-kv_z-\delta)/2}G^{+*}\left(x_1-x_A,z_1-z_A\right) \tag{105}$$

the momentum induced out of resonance disappears and the following result is obtained for the temporal beam splitter (rectangular pulse in the time domain)

[11] A better approximation is to neglect the dispersion of the first-order Doppler shift only in the envelope and to write $a^{(0)}(x_1,y_1,z_1-\frac{\hbar k}{Mv_{0x}}(x_1-x_A),t_1)$.

$$b^{(1)}(\vec{r}_1, t_1) = i\Omega_{ba}^{+} e^{-i\varphi^{+}} e^{ikz_1} U_0^{+*}(\vec{r}_1 - \vec{r}_A)$$
$$\int d^3 p e^{-i(\omega_{ba}+kv_z+\delta)t_1} e^{-i(\Delta-kv_z-\delta)t_A}$$
$$\frac{\sin((\Delta - kv_z - \delta)\tau/2)}{(\Delta - kv_z - \delta)/2} \langle \vec{r}_1|\vec{p}\rangle \langle \vec{p}, a|U_0(t_1, t_0)|\Psi^{(0)}(t_0)\rangle \quad (106)$$

and if we neglect the dispersion of the splitter[12]

$$b^{(1)}(\vec{r}_1, t_1) = i\left(\Omega_{ba}^{+}\tau\right) e^{-i(\omega t_1 - kz_1 + \varphi^{+})} U_0^{+*}(\vec{r}_1 - \vec{r}_A) e^{i(\Delta - kv_{0z} - \delta)(t_1 - t_A)}$$
$$\frac{\sin((\Delta - kv_{0z} - \delta)\tau/2)}{(\Delta - kv_{0z} - \delta)\tau/2} a^{(0)}(\vec{r}_1, t_1). \quad (107)$$

In both cases the incident wave packet given by the $ABCD\xi$ theorem [7, 8, 19]

$$a^{(0)}(\vec{r}_1, t_1) = \langle \vec{r}_1, a|\Psi^{(0)}(t_1)\rangle$$
$$= \langle \vec{r}_1, a|U_0(t_1, t_0)|\Psi^{(0)}(t_0)\rangle$$
$$= e^{iS_a(t_1,t_0)/\hbar} e^{i\vec{p}_a(t_1)\cdot(\vec{r}_1 - \vec{r}_c(t_1))/\hbar} \mathcal{F}(\vec{r}_1 - \vec{r}_c(t_1), X(t_1), Y(t_1)) \quad (108)$$

is multiplied by space-dependent Gaussians that we shall assume either centered about the same position as the wave packet or broad enough to be ignored. When multiplied by these, the wave-packet envelope will keep its Gaussian or Hermite-Gauss character. In all cases we shall write the multiplication factor introduced by the splitter as:

$$M_{ba} e^{-i(\omega t_1 - kz_1 + \varphi^{+})} e^{i(\Delta - kv_{0z} - \delta)(t_1 - t_A)} \quad (109)$$

with

$$M_{ba} = i\left(\Omega_{ba}^{+}\tau\right) U_0^{+*} \frac{\sin((\Delta - kv_{0z} - \delta)\tau/2)}{(\Delta - kv_{0z} - \delta)\tau/2} \quad (110)$$

or

$$M_{ba} e^{-i(\omega t_1 - kz_1 + \varphi^{+})} e^{ik_{0x}(x_1 - x_A)} \quad (111)$$

with

$$M_{ba} = i\sqrt{\pi}\left(\frac{\Omega_{ba}^{+} w_0}{v_{0x}}\right) G^{+*} \exp\left[-\frac{(k_{0x} w_0)^2}{4L^{+*}}\right] \quad (112)$$

The same phase factors also appear in the strong field theory of beam splitters ([1, 11]).

[12] A better approximation is to neglect the dispersion of the first-order Doppler shift only in the envelope and to write $a^{(0)}(x_1, y_1, z_1 - \frac{\hbar k}{M}(t_1 - t_A), t_1)$.

In order to apply the ABCDξ theorem, space-dependent phase factors like $e^{ik_{0x}(x_1-x_A)}$ or e^{ikz_1} will be rewritten as:

$$e^{ik_{0x}(x_1-x_c(t_1))}e^{ik_{0x}(x_c(t_1)-x_A)} \tag{113}$$

or

$$e^{ik(z_1-z_c(t_1))}e^{ikz_c(t_1)} \tag{114}$$

In the temporal beam splitter case, the excited state amplitude at $(\vec{r},t)$ will thus be:

$$\begin{aligned} b^{(1)}\left(\vec{r},t\right) &= \int d^3r_1 \langle \vec{r},b|U_0(t,t_1)|\vec{r}_1,b\rangle b^{(1)}(\vec{r}_1,t_1) \\ &= \int d^3r_1 \mathcal{K}_\alpha(\vec{r},\vec{r}_1,t,t_1) b^{(1)}(\vec{r}_1,t_1) \\ &= M_{ba} e^{-i(\omega t_1+\varphi^+)} e^{iS_b(t,t_1)/\hbar} e^{i\vec{p}_b(t)\cdot\left(\vec{r}-\vec{r}_c(t)\right)/\hbar} \\ &\quad e^{i(\Delta-kv_{0z}-\delta)(t_1-t_A)} e^{ikz_c(t_1)} \\ &\quad e^{iS_a(t_1,t_0)/\hbar} \mathcal{F}(\vec{r}-\vec{r}_c(t),X(t),Y(t)) \end{aligned} \tag{115}$$

or with the choice $t_1 = t_A$

$$\begin{aligned} b^{(1)}(\vec{r},t) = M_{ba} & e^{iS_b(t,t_A)/\hbar} e^{i\vec{p}_b(t)\cdot\left(\vec{r}-\vec{r}_c(t)\right)/\hbar} e^{-i(\omega t_A - kz_c(t_A)+\varphi^+)} \\ & e^{iS_a(t_A,t_0)/\hbar} \mathcal{F}(\vec{r}-\vec{r}_c(t),X(t),Y(t)) \end{aligned} \tag{116}$$

with

$$\vec{p}_b(t_A) = \vec{p}_a(t_A) + \hbar k\hat{z}. \tag{117}$$

In the spatial beam splitter case:

$$\begin{aligned} b^{(1)}(\vec{r},t) &= \int d^3r_1 \langle \vec{r},b|U_0(t,t_1)|\vec{r}_1,b\rangle b^{(1)}(\vec{r}_1,t_1) \\ &= \int d^3r_1 \mathcal{K}_\alpha(\vec{r},\vec{r}_1,t,t_1) b^{(1)}(\vec{r}_1,t_1) \\ &= M_{ba} e^{-i(\omega t_1+\varphi^+)} e^{iS_b(t,t_1)/\hbar} e^{i\vec{p}_b(t)\cdot(\vec{r}-\vec{r}_c(t))/\hbar} e^{ik_{0x}(x_c(t_1)-x_A)} \\ &\quad e^{ikz_c(t_1)} e^{iS_a(t_1,t_0)/\hbar} \mathcal{F}(\vec{r}-\vec{r}_c(t),X(t),Y(t)) \end{aligned} \tag{118}$$

or with the choice of $t_1 = t'_A$ such that $x_c(t'_A) = x_A$

$$\begin{aligned} b^{(1)}(\vec{r},t) = M_{ba} & e^{iS_b(t,t'_A)/\hbar} e^{i\vec{p}_b(t)\cdot(\vec{r}-\vec{r}_c(t))/\hbar} e^{-i(\omega t'_A - kz_c(t'_A)+\varphi^+)} \\ & e^{iS_a(t'_A,t_0)/\hbar} \mathcal{F}(\vec{r}-\vec{r}_c(t),X(t),Y(t)) \end{aligned} \tag{119}$$

which is the same formula as in the previous case but now with

$$\vec{p}_b(t'_A) = \vec{p}_a(t'_A) + \hbar k_{0x}\hat{x} + \hbar k\hat{z} \tag{120}$$

i.e. with an additional momentum in the longitudinal direction and proportional to the detuning.

APPENDIX E

Case of Two-Photon Transitions

In this appendix, we extend the results of the first-order amplitude calculations obtained in the previous appendix for the one-photon case to two-photon transitions. We shall not consider the temporal dependence of the e.m. field, which leads to formulas similar to the one-photon case and, for simplicity, we assume also equal frequencies for both fields. The formulas are easily generalized to Raman transitions and fields. The example treated here corresponds to Doppler-free two-photon Ramsey fringes with counterpropagating fields in a cascade configuration (with an application to hydrogen in mind). The matrix element of the interaction Hamiltonian is given by:

$$V_{ba}(\vec{r},t) = -\hbar\Omega_{\text{eff}} e^{i(2\omega t+\varphi^{+}+\varphi^{-})} U^{+}(\vec{r}-\vec{r}_1) U^{-}(\vec{r}-\vec{r}_1) + c.c. + (+ \leftrightarrow -) \tag{121}$$

where Ω_{eff} is an effective Rabi frequency and

$$\begin{aligned}
W(\vec{r}) &= U^{+}(\vec{r})U^{-}(\vec{r}) \\
&= L^{+}(z)L^{-}(z)\exp\left[-\frac{L^{+}(z)+L^{-}(z)}{w_0^2}(x^2+y^2)\right] \\
&= \frac{w_0^2}{w^2(z)}\exp\left[-\frac{2(x^2+y^2)}{w^2(z)}\right] \\
&= \frac{w_0^2}{8\pi}\int dk_x dk_y e^{i(k_x x+k_y y)}\exp\left[-\frac{\left(k_x^2+k_y^2\right)w^2(z)}{8}\right] \\
&= \frac{w_0^3}{4\,(2\pi)^{3/2}}\int d^3k\,\frac{k}{k_\perp}e^{i(k_x x+k_y y+k_z z)} \\
&\quad \exp\left[-\frac{k_\perp^2 w_0^2}{8}\right]\exp\left[-k_z^2\frac{k^2 w_0^2}{2k_\perp^2}\right].
\end{aligned} \tag{122}$$

In the case of copropagating fields U^- is replaced by U^+ and there is an additional e^{-2ikz} factor. For Raman transitions U^- would be replaced by U^{-*} with an additional $e^{-i(k_1+k_2)z}$ factor.

Note that now $k_{\text{eff}}^2 = k_x^2 + k_y^2 + k_z^2 \neq k^2$

$$\begin{aligned}
V_{ba}(\vec{k},t) = &-\hbar\Omega_{\text{eff}} e^{i(2\omega t+\varphi^{+}+\varphi^{-})}\frac{w_0^3}{4}\frac{k}{k_\perp}\exp\left[-\frac{k_\perp^2 w_0^2}{8}\right]\exp\left[-k_z^2\frac{k^2 w_0^2}{2k_\perp^2}\right] \\
&+ c.c. + (+ \leftrightarrow -)
\end{aligned} \tag{123}$$

and we write the atomic energy factor as (γ_b is the upper state decay rate)

$$e^{i[E_b(\vec{p}+\hbar\vec{k}_{\text{eff}})-E_a(\vec{p})-i\hbar\gamma_b/2](t'-t)/\hbar} = e^{i[\omega_{ba}+\vec{k}_{\text{eff}}\cdot\vec{v}+\hbar k_{\text{eff}}^2/2M-i\gamma_b/2](t'-t)}. \tag{124}$$

With the rotating-wave approximation

$$\begin{aligned} b^{(1)}(\vec{r},t) = &\frac{1}{i\hbar}\int_{-\infty}^{t}dt'\int\frac{d^3p}{(2\pi\hbar)^{3/2}}\int\frac{d^3k}{(2\pi)^{3/2}}V_{ba}(\vec{k},t')e^{i\vec{k}\cdot(\vec{r}-\vec{r}_1)}\\ &e^{i[E_b(\vec{p}+\hbar\vec{k})-E_a(\vec{p})-i\gamma_b/2](t'-t)/\hbar}\\ &e^{i[\vec{p}\cdot(\vec{r}-\vec{r}_0)-E_a(\vec{p})(t-t_0)]/\hbar}\langle a,\vec{p}|\Psi^{(0)}\rangle \qquad (125)\\ = &\, i\Omega_{\text{eff}}e^{-i(2\omega t+\varphi^++\varphi^-)}\frac{w_0^3}{4(2\pi)^{3/2}}\int\frac{d^3p}{(2\pi\hbar)^{3/2}} \qquad (126)\\ &\int d^3k\frac{k}{k_\perp}e^{i\vec{k}\cdot(\vec{r}-\vec{r}_1)}\exp\left[-\frac{k_\perp^2w_0^2}{8}\right]\exp\left[-k_z^2\frac{k^2w_0^2}{2k_\perp^2}\right] \qquad (127)\\ &\int_{-\infty}^{t}dt'e^{-i[2\omega-\omega_{ba}-\vec{k}_{\text{eff}}\cdot\vec{v}-\frac{\hbar k_{\text{eff}}^2}{2M}+i\gamma_b/2](t'-t)} \qquad (128)\\ &e^{i[\vec{p}\cdot(\vec{r}-\vec{r}_0)-E_a(\vec{p})(t-t_0)]/\hbar}\left\langle a,\vec{p}|\Psi^{(0)}\right\rangle \qquad (129) \end{aligned}$$

If we neglect the longitudinal recoil term $\hbar k_z^2/2M$, then the k_z integral has a simple expression

$$\begin{aligned} w_0\int\frac{dk_z}{(2\pi)^{1/2}}\frac{k}{k_\perp}\exp\left[-k_z^2\frac{k^2w_0^2}{2k_\perp^2}\right]e^{ik_z(z-z_1-v_z(t-t'))}\\ = \exp\left[-(z-z_1-v_z(t-t'))^2\frac{k_\perp^2}{2k^2w_0^2}\right] \end{aligned} \tag{130}$$

and

$$\begin{aligned} b^{(1)}(\vec{r},t) = &\, i\Omega_{\text{eff}}e^{-i(2\omega t+\varphi^++\varphi^-)}\frac{w_0^2}{8\pi}\int\frac{d^3p}{(2\pi\hbar)^{3/2}}\int dk_xdk_ye^{i\vec{k}_\perp\cdot(\vec{r}-\vec{r}_1)}\\ &\int_{-\infty}^{t}dt'e^{-i[2\omega-\omega_{ba}-\vec{k}_\perp\cdot\vec{v}-\hbar k_\perp^2/2M+i\gamma_b/2](t'-t)}\\ &\exp\left[-\frac{k_\perp^2w_0^2}{8}\right]\exp\left[-(z-z_1-v_z(t-t'))^2\frac{k_\perp^2}{2k^2w_0^2}\right]\\ &e^{i[\vec{p}\cdot(\vec{r}-\vec{r}_0)-E_a(\vec{p})(t-t_0)]/\hbar}\left\langle a,\vec{p}|\Psi^{(0)}\right\rangle \end{aligned} \tag{131}$$

or

$$b^{(1)}(\vec{r},t) = i\Omega_{\text{eff}} e^{-i(2\omega t+\varphi^+ +\varphi^-)} \frac{w_0^2}{8\pi} \int \frac{d^3p}{(2\pi\hbar)^{3/2}} \int dk_x dk_y e^{i\vec{k}_\perp\cdot(\vec{r}-\vec{r}_1)}$$
$$\int_{-\infty}^{t} dt' e^{-i[2\omega-\omega_{ba}-\vec{k}_\perp\cdot\vec{v}-\hbar k_\perp^2/2M+i\gamma_b/2](t'-t)}$$
$$\exp\left[-\frac{k_\perp^2}{8} w^2(z-z_1-v_z(t-t'))\right]$$
$$e^{i[\vec{p}\cdot(\vec{r}-\vec{r}_0)-E_a(\vec{p})(t-t_0)]/\hbar}\langle a,\vec{p}|\Psi^{(0)}\rangle \tag{132}$$

with

$$w^2(z) = w_0^2\left[1+4\frac{z^2}{k^2w_0^4}\right]. \tag{133}$$

We could let $t \to +\infty$ for a field bounded in space as in Appendic D to introduce a δ function expressing energy conservation but, for the illustration we prefer here to proceed with the exact calculation for finite times. If the recoil shift is small enough, we may use a first-order expansion

$$b^{(1)}(\vec{r},t) = i\Omega_{\text{eff}} e^{-i(2\omega t+\varphi^+ +\varphi^-)} \int \frac{d^3p}{(2\pi\hbar)^{3/2}} e^{i[\vec{p}\cdot(\vec{r}-\vec{r}_0)-E_a(\vec{p})(t-t_0)]/\hbar}\langle a,\vec{p}|\Psi^{(0)}\rangle$$
$$\int_{-\infty}^{t} dt' e^{i[2\omega-\omega_{ba}+i\gamma_b/2](t-t')}\Big[W(\vec{r}-\vec{r}_1-\vec{v}(t-t'))$$
$$+i\frac{\hbar}{2M}(t-t')\nabla_\perp^2 W(\vec{r}-\vec{r}_1-\vec{v}(t-t'))\Big], \tag{134}$$

where

$$\int_{-\infty}^{t} dt' e^{i[2\omega-\omega_{ba}+i\gamma_b/2](t-t')} W(\vec{r}-\vec{r}_1-\vec{v}(t-t'))$$
$$= \int_0^{+\infty} d\tau \frac{w_0^2}{w^2(z-z_1-v_z\tau)}$$
$$\exp\left[-\frac{2\left[(x-x_1-v_x\tau)^2+(y-y_1-v_y\tau)^2\right]}{w^2(z-z_1-v_z\tau)}\right] e^{i[2\omega-\omega_{ba}+i\gamma_b/2]\tau}. \tag{135}$$

If the longitudinal transit-time broadening is neglected, this integral is easily calculated as in the one-photon case. For $v_y = 0$ and $\gamma_b = 0$ and with $\Delta = 2\omega - \omega_{ba}$, one finds

$$b^{(1)}(\vec{r},t) = i\frac{\sqrt{\pi}}{2\sqrt{2}}\Omega_{\text{eff}}\frac{w}{v_x}e^{-i(2\omega t+\varphi^+ +\varphi^-)}\exp\left(-\frac{w^2\Delta^2}{8v_x^2}\right)$$
$$e^{i\frac{\Delta(x-x_1)}{v_x}}\left[1+\text{erf}\left(\frac{i}{2\sqrt{2}}\frac{w}{v_x}\Delta+\sqrt{2}\frac{(x-x_1)}{w}\right)\right]$$
$$\left[1+i\frac{\hbar}{Mv_x w}\left[\Delta\frac{w}{v_x}-\frac{1}{8}\left(\Delta\frac{w}{v_x}\right)^3\right]\right]a^{(0)}(\vec{r},t) \qquad (136)$$

where the last two terms in the final bracket give the recoil correction to the lineshape (only terms leading to a shift have been conserved). These terms scale with the ratio of the de Broglie wave to the laser beam radius. When the wave packet has left the interaction zone the error function $\longrightarrow 1$. Here again we find an additional momentum, communicated to the atom wave, proportional to the detuning, which will lead to the formation of Ramsey fringes, which can be seen in the crossed term of the modulus squared $b^{(1)}(\vec{r},t)b^{(1)*}(\vec{r},t)$ corresponding to field zone centers x_1 and x_2.

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