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Atomic Interferometry in Gravitational Fields: Influence of Gravitation on the Beam Splitter

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The laser induced splitting of atomic beams in the presence of a gravitational field is analyzed. In the frame of a quasiclassical approximation, the motion of the atomic beam through a laser region with rectangular profile is calculated. Beside the usual beam splitting due to the atom–laser interaction, an additional splitting occurs due to the anomalous effective interaction with the gravitational field. In a first order approximation in the gravitational acceleration, the outcome of an atom interferometry experiment is given, which includes the various corrections owing to the gravitational modification of the beam splitting process.

KEY WORDS : Anomalous effective interaction

1. INTRODUCTION

Interferometry using laser beam splitters [1–6] has proved to be a very successful tool in probing the interaction of quantum objects with gravitational and inertial fields. The accuracy of these devices makes it necessary to consider also the relativistic corrections of their phase shifts due to acceleration and rotation as well as to treat the influence of spin [7,10].

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The theoretical importance of quantum interference experiments for the development of a theory of gravity which is appropriate to the quantum description of matter has been stressed by Hehl [8,9].

The main part of these devices is the beam splitter, which consists of a laser beam interacting with the atoms or the molecules. Since this interaction takes place in the presence of external gravitational and inertial fields, it is expected that there will be a modification of the atom beam splitting process as compared to the splitting in the case of vanishing gravitational and inertial fields. It is necessary to give an appropriate description of the beam splitting process and to calculate the corresponding corrections in order to give a correct interpretation of the experimental results.

There are two kinds of beam splitters and correspondingly two kinds of atom interferometers: one uses a stationary interaction geometry with time-independent laser beams while the other uses laser pulses. The first case is a time-independent quantum mechanical problem while the second is time-dependent. The latter problem has an exact solution, even in the case of an additional gravitational field [11,12]. Here we want to give a quantum mechanical description of the beam splitting process in the stationary case.

We treat the beam splitting process as the scattering of an atomic beam by a periodic potential which is given by the laser beam. In our case, an additional gravitational potential is present. We proceed in the same manner as in usual calculations of the transmission of a plane wave through a potential barrier. However, in our case the problem is more complicated owing to the two-level structure of the quantum state, the periodic structure of the scattering potential, and the additional gravitational potential.

We assume that the periodic potential has a rectangular profile (see Fig. 1). The laser region is bounded by two parallel planes Σ and Σ' possessing a common normal $\boldsymbol{n}$. We calculate the intensity of the outgoing atomic beam as a function of the incoming beam. There we can additionally distinguish between the cases where the incoming atoms are in the ground state or in the excited state.

In order to describe the beam splitters in the spatial interaction geometry we restrict ourselves to stationary states which are described by the stationary two-level Schrödinger equation,

$$E\Psi(\boldsymbol{x}) = \left(\frac{\widehat{\boldsymbol{p}}^2}{2m} - \frac{1}{2}\hbar\omega\sigma^3 + H_{\mathrm{a}} + H_{\mathrm{dip}}(\boldsymbol{x}) + H_{\mathrm{int}}(\boldsymbol{x})\right)\Psi(\boldsymbol{x}). \qquad (1)$$

Here $\Psi = \binom{\Psi_b}{\Psi_a}$ is a two-component wave function, σ^3 is the third Pauli-

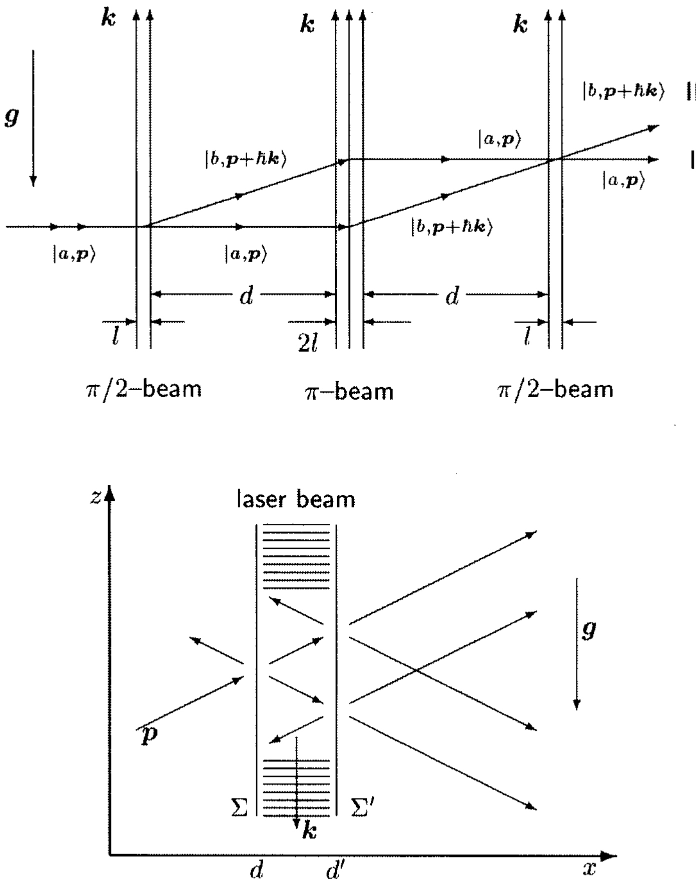

Figure 1. Upper: Geometry of a simple atom beam interferometer; see [13]. An atomic beam with atoms in the ground state with momentum $\boldsymbol{p}$, $|a, \boldsymbol{p}\rangle$, is split during the interaction with a first laser beam into the original state and the excited state with momentum $\boldsymbol{p} + \hbar\boldsymbol{k}$. The other two laser beams serve as mirror and analyzer. The interference pattern is observed by counting the excited atoms leaving the analyzer. Lower: Geometry of the beam splitters. Atomic plane waves with momentum $\boldsymbol{p}$ enter the laser region between the surfaces Σ and Σ'. These surfaces have the x coordinates d and $d' = d + l$. The wave vector of the laser beam $\boldsymbol{k}$ is transferred with a certain probability to the atomic wave so that the transmitted atomic wave is split into two waves.

matrix, and

$$H_{\mathrm{a}} = \begin{pmatrix} E_b & 0 \\ 0 & E_a \end{pmatrix} = \frac{1}{2}(E_a + E_b) + \frac{1}{2}\hbar\omega_{ba}\sigma^3, \tag{2}$$

$$H_{\mathrm{dip}}(\boldsymbol{x}) = -\hbar\Omega_{ba}\begin{pmatrix} 0 & e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \\ e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} & 0 \end{pmatrix}, \tag{3}$$

$$H_{\mathrm{int}}(\boldsymbol{x}) = -m\boldsymbol{g}\cdot\boldsymbol{x}, \tag{4}$$

with $\omega_{ba} := (E_b - E_a)/\hbar$, where E_b and E_a are the upper and lower energy levels of the atom, m is the mass of the atom and $\Omega_{ba} = (e/2\hbar)|\langle a|\mu E|b\rangle|$ is the Rabi frequency. ω and $\boldsymbol{k}$ are the frequency and the wave vector of the laser beam, respectively. We neglect any relaxation effect. We have already removed any time-dependence from the dipole interaction term — compare [11].

2. WKB SOLUTION IN THE LASER REGION

In two-component notation (1) is given by

$$E'\Psi = \begin{pmatrix} -(\hbar^2/2m)\nabla^2 - \hbar\Delta - m\boldsymbol{g}\cdot\boldsymbol{x} & -\hbar\Omega_{ba}e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \\ -\hbar\Omega_{ba}e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} & -(\hbar^2/2m)\nabla^2 - m\boldsymbol{g}\cdot\boldsymbol{x} \end{pmatrix}\Psi, \tag{5}$$

where[4] $\Delta = \omega - \omega_{ba}$ is the detuning. We first define

$$\psi(\boldsymbol{x}) = \begin{pmatrix} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} & 0 \\ 0 & 1 \end{pmatrix}\Psi(\boldsymbol{x}), \tag{6}$$

which means that the upper state acquires an extra momentum $\boldsymbol{k}$. Then we make the WKB-ansatz

$$\psi(\boldsymbol{x}) = e^{-(i/\hbar)S(\boldsymbol{x})}a(\boldsymbol{x}) \tag{7}$$

with a real valued scalar phase $S(\boldsymbol{x})$. We assume that $a(\boldsymbol{x})$ and $\boldsymbol{p} = -\nabla S$ are only slowly varying. Insertion of the ansatz (7) gives

$$0 = \begin{pmatrix} ((\boldsymbol{p} + \hbar\boldsymbol{k})^2/2m) - E' - \hbar\Delta - m\boldsymbol{g}\cdot\boldsymbol{x} & -\hbar\Omega_{ba} \\ -\hbar\Omega_{ba} & (\boldsymbol{p}^2/2m) - E' - m\boldsymbol{g}\cdot\boldsymbol{x} \end{pmatrix} a. \tag{8}$$

4 The Laplacian is denoted by ∇^2.

The Hamilton–Jacobi equation is given by the solvability condition

$$\hbar^2 \Omega_{ba}^2 = \left(\frac{(\boldsymbol{p} + \hbar \boldsymbol{k})^2}{2m} - E' - \hbar\Delta - m\boldsymbol{g} \cdot \boldsymbol{x} \right) \left(\frac{\boldsymbol{p}^2}{2m} - E' - m\boldsymbol{g} \cdot \boldsymbol{x} \right). \tag{9}$$

For the momentum $\boldsymbol{p}$, this is a fourth-order equation, that is, if e.g. p_y and p_z are given then there will be four solutions for the component p_x. Therefore, in general the solution consists of four waves and the mass shell has two disconnected components, called branches. (A thorough discussion of the dispersion surface will is in [14].)

We solve this equation for constant energy E' and get two Hamilton–Jacobi equations or two dispersion surfaces with the corresponding solutions $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$:

$$E' = \frac{\boldsymbol{p}_{1,2}^2}{2m} - m\boldsymbol{g} \cdot \boldsymbol{x} + \hbar\Omega_{ba}(y(\boldsymbol{p}_{1,2}) \mp \sqrt{1 + y^2(\boldsymbol{p}_{1,2})}) \tag{10}$$

with

$$y(\boldsymbol{p}) := \frac{1}{2\hbar\Omega_{ba}} \left(\frac{(\boldsymbol{p} + \hbar\boldsymbol{k})^2}{2m} - \frac{\boldsymbol{p}^2}{2m} - \hbar\Delta \right). \tag{11}$$

Note that y does not depend explicitly on the position[5] . It depends on $\boldsymbol{p}$ through the combination $\boldsymbol{p} \cdot \boldsymbol{k}$. Consequently, it is most appropriate to choose one of the momentum components which have to be prescribed, to be that component in the direction of $\boldsymbol{k}$. In this case we have $y(\boldsymbol{p}_1) = y(\boldsymbol{p}_2) = y(\boldsymbol{p})$. This function y is introduced in analogy to the theory of dynamical neutron diffraction which has many features in common with our theory (see Ref. 10).

The two Hamilton–Jacobi equations can be rewritten as ($\boldsymbol{p}_t = (p_y, p_z)$)

$$\mathcal{E}_{1,2}(\boldsymbol{p}_t) = \frac{p_x^2}{2m} - m\boldsymbol{g} \cdot \boldsymbol{x} \tag{13}$$

[5] This is no longer true for a coupling to a rotating frame. In this case the coefficient y reads

$$\begin{aligned} y(\boldsymbol{x}, \boldsymbol{p}) &= \frac{1}{2\hbar\Omega_{ba}} \left(H(\boldsymbol{x}, \boldsymbol{p} + \hbar\boldsymbol{k}) - H(\boldsymbol{x}, \boldsymbol{p}) - \hbar\Delta \right) \\ &= \frac{1}{2\Omega_{ba}} \left(\left(\frac{1}{m}\boldsymbol{p} - \boldsymbol{x} \times \Omega \right) \cdot \boldsymbol{k} + \frac{\hbar}{2m}\boldsymbol{k}^2 - \Delta \right), \end{aligned} \tag{12}$$

so that there is an additional $\boldsymbol{x}$-dependence which has to be taken into account and leads to a correction of the Sagnac effect.

with

$$\mathcal{E}_{1,2}(\boldsymbol{p}_t) = E' - \frac{\boldsymbol{p}_\perp^{\;2}}{2m} - \hbar\Omega_{ba}\left(y(\boldsymbol{p}) \mp \sqrt{1 + y^2(\boldsymbol{p})}\,\right). \tag{14}$$

In the following we will neglect reflected waves.

We first discuss some general features of the classical equations of motion. These equations are given by

$$\frac{d}{dt}\boldsymbol{x}_{1,2} = \frac{\boldsymbol{p}_{1,2}}{m} + \frac{\hbar\boldsymbol{k}}{m}A_{\mp}(\boldsymbol{p}), \qquad \frac{d}{dt}\boldsymbol{p}_{1,2} = m\boldsymbol{g}, \tag{15}$$

with

$$A_{\pm}(\boldsymbol{p}) := \frac{1}{2}\left(1 \pm \frac{y(\boldsymbol{p})}{\sqrt{1 + y^2(\boldsymbol{p})}}\right). \tag{16}$$

The solution of (15) is

$$\boldsymbol{p}_{1,2}(t) = \boldsymbol{p}_{1,2}^0 + m\boldsymbol{g}t \tag{17}$$

where the subscript 1,2 for $\boldsymbol{p}_{1,2}^0$ indicates that for two given components of the initial momentum there are two solutions for the third component due to the two Hamilton–Jacobi equations. The equation for $\boldsymbol{x}$ has to be differentiated once more:

$$\frac{d^2}{dt^2}\boldsymbol{x}_{1,2} = \boldsymbol{g} \mp \frac{\hbar\boldsymbol{k}}{4m\Omega_{ba}}\,\frac{(\boldsymbol{g}\cdot\boldsymbol{k})}{(1 + y^2(\boldsymbol{p}))^{3/2}}\,. \tag{18}$$

This equation of motion describes that each branch is subject to another effective acceleration induced by the gravitational force. The second term can be varied by manipulating the Rabi frequency and the wave vector. If we introduce the effective mass tensor

$$\left(\frac{1}{m_{1,2}^*}\right)^{ij} := \left(\frac{\partial^2 H}{\partial p_i \partial p_j}\right)_{p=p_{1,2}}, \tag{19}$$

then (18) has the form

$$\frac{d^2}{dt^2}x_{1,2}^i = \left(\frac{1}{m_{1,2}^*}\right)^{ij} g_j\,. \tag{20}$$

The effective mass tensor, which still depends on the momentum p, is responsible for two different accelerations of the atom which leads to two different atomic trajectories inside the laser beam region.

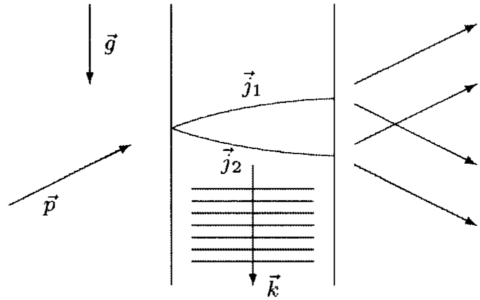

Figure 2. Anomalous interaction of the two-level system with an external force: after the splitting of the incoming atomic beam one part of the beam accelerates differently from the other, though both are subject to the same force.

With the solution for the momentum, the equation for the path can be explicitly integrated. However, in order to display compact results, we restrict ourselves to first order in the gravitational acceleration and specialize to the case $\boldsymbol{k} = k\boldsymbol{e}_z$, $\boldsymbol{g} = g\boldsymbol{e}_z$, $y_0 = 0$, and $p_{0y} = 0$.

$$x_{1,2}(t) = x_0 + \frac{1}{m}\sqrt{2m\varepsilon_{1,2}(p_{0z})}\left(1 + \frac{mgz}{2\varepsilon_{1,2}(p_{0z})}\right)t, \tag{21}$$

$$\begin{aligned} z_{1,2}(t) &= z_0 + \frac{1}{m}\left(p_{0z} + \hbar k A_{\pm}(p_{0z})\right)t \\ &\quad + \frac{1}{2}\left(1 \mp \frac{\hbar k^2}{4m\Omega_{ba}}\frac{1}{(1 + y^2(p_{0z}))^{3/2}}\right)gt^2. \end{aligned} \tag{22}$$

The second term describes the usual beam splitting of the atomic beam inside the laser region. The last term is a gravitational modification of the beam splitting process.

We have to look for the phase and the corresponding momenta depending on the spatial coordinates only. We choose again $x = x_0$ as the boundary along which the phases $S_{1,2}(x, z)$ are given. We get for the phases $(p_{0z}(z) = p_z(x_0, z))$,

$$\begin{aligned} S_{1,2}(x, z) &= -p_{0z}(z)z - \sqrt{2m(\varepsilon_{1,2}(p_{0z}(z)) + mgz)}\,(x - x_0) \\ &\quad + \frac{gmp_{0z}(z)}{4\varepsilon_{1,2}(p_{0z}(z))}(x - x_0)^2 \end{aligned} \tag{23}$$

and the corresponding momenta

$$p_{x1,2}(x, z) = \sqrt{2m\,(\varepsilon_{1,2}(p_{0z}(z)) + mgz)} - \frac{gmp_{0z}(z)}{2\varepsilon_{1,2}(p_{0z}(z))}(x - x_0) \tag{24}$$

$$p_{z1,2}(x,z) = p_{0z}(z) + m^2 g \frac{x - x_0}{\sqrt{2m\varepsilon_{1,2}(p_{0z}(z))}} . \tag{25}$$

These momenta solve the Hamilton–Jacobi equation to first order in g. Note that there is a splitting of the momentum in the z-direction, which is not present in the case of vanishing gravitational interaction.

The normalized amplitudes can be found from eq. (8) which gives the solution

$$\psi(x) = \alpha_1 \begin{pmatrix} \sqrt{A_+(\boldsymbol{p})} \\ -\sqrt{A_-(\boldsymbol{p})} \end{pmatrix} e^{-iS_1(\boldsymbol{X})} + \alpha_2 \begin{pmatrix} \sqrt{A_-(\boldsymbol{p})} \\ \sqrt{A_+(\boldsymbol{p})} \end{pmatrix} e^{-iS_2(\boldsymbol{X})} . \tag{26}$$

In the case of vanishing laser intensity ($\Omega_{ba} = 0$) we have $A_+(\boldsymbol{p}) = 1$ and $A_-(\boldsymbol{p}) = 0$ and therefore[6]

$$\psi^0(x) = \alpha_1^0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-iS_1^0(\boldsymbol{X})} + \alpha_2^0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-iS_2^0(\boldsymbol{X})} . \tag{27}$$

We also have in this case from (10) $E' = (1/2m)(p_{x1}^2 + p_{z1}^2) - mgz$ so that for $z = 0$ the E' is just the kinetic energy of a gound state atom.

3. THE BEAM SPLITTER IN A GRAVITATIONAL FIELD

We turn to the description of the beam splitter in the presence of a gravitational field. For doing so, we first take an atomic beam with a certain phase with boundary condition $S^0(x = 0, z) = \text{const}$. This atomic beam with a given phase propagates in a laser free region to the entrance surface at $x = d$. There it enters the laser region, propagates through the laser region, and leaves it. At the entrance and exit surfaces, jump conditions have to be used, and during the propagation the calculated phase gives the solution. Since at each point we can assign a wave vector to the wave function, we can use the jump conditions derived above locally.

We proceed in several steps. We consider how the wave function enters the laser region at the entrance surface at $x = d$, propagates to the exit surface at $x = d'$ and leaves the laser region. At the end we combine all these steps and get a relation between the wave function entering the laser region and the wave function leaving the laser region.

Step 1: The incoming state in front of the beam splitter is given by (27).

Step 2: The state behind the entrance surface of the beam splitter is (26). These two wave functions are connected by jump conditions: (i) the

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[6] A superscript 0 refers to quantities in the laser-free region.

tangential part of the momentum is continuous along the boundary, and (ii) the state is continuous over the entrance surface.

The first condition, together with the constancy of the energy, gives $p_x(d,z)$ as a function of $p_z^0(d,z)$:

$$p_{x1,2}(d,z) = p_{x1,2}^0(d,z) + \frac{m\hbar\Omega_{ba}}{p_{x1,2}^0(d,z)}\left(y(p_z^0(d,z)) \mp \sqrt{1+y^2(p_z^0(d,z))}\,\right), \quad (28)$$

where we have made another approximation $\hbar\Omega_{ba} \ll (p_{x1,2}^0(d,z))^2/2m$, which means that the kinetic energy of the atoms is large compared to the interaction energy of the laser beam.

The continuity of the wave function, that is, the equality of (26) and (27) at $x = d$, relates the amplitudes inside the laser region to the amplitudes outside:

$$\alpha_1 = \alpha_1^0\sqrt{A_+(p_z(d,z))}e^{is_{11}} - \alpha_2^0\sqrt{A_-(p_z(d,z))}\,e^{is_{21}}, \quad (29)$$

$$\alpha_2 = \alpha_1^0\sqrt{A_-(p_z(d,z))}e^{is_{12}} + \alpha_2^0\sqrt{A_+(p_z(d,z))}\,e^{is_{22}}, \quad (30)$$

and the corresponding inverse relations, where we have introduced $S_i^0(d,z) - S_j(d,z) =: \hbar s_{ij}$ $(i,j = 1,2)$. Consequently, the state behind the entrance boundary is given by (26) with α_1 and α_2 from (29) and (30).

Step 3: The state at the exit surface at $x = d'$ inside the laser region is

$$\psi(d',z) = \psi_1(d',z) + \psi_2(d',z),$$

$$\psi_1(d',z) = \left(\alpha_1^0\sqrt{A_+(p_z(d',z))}\,e^{is_{11}} - \alpha_2^0\sqrt{A_-(p_z(d',z))}\,e^{is_{21}}\right)\begin{pmatrix}\sqrt{A_+(p_z(d',z))}\\ -\sqrt{A_-(p_z(d',z))}\end{pmatrix}e^{iS_1(d',z)}, \quad (32)$$

$$\psi_2(d',z) = \left(\alpha_1^0\sqrt{A_-(p_z(d',z))}\,e^{is_{12}} + \alpha_2^0\sqrt{A_+(p_z(d',z))}\,e^{is_{22}}\right)\begin{pmatrix}\sqrt{A_-(p_z(d',z))}\\ \sqrt{A_+(p_z(d',z))}\end{pmatrix}e^{iS_2(d',z)}. \quad (33)$$

It is now essential to note that the two states no longer possess the same $p_z(d',z)$ as it is the case without gravity. These momenta are given by (24),(25)

$$p_{x1,2}(d',z) = \sqrt{2m(\varepsilon_{1,2}(p_z(d)) + mgz)}, \quad (34)$$

$$p_{z1,2}(d',z) = p_z(d,z) + \frac{m^2g}{\sqrt{2m\varepsilon_{1,2}(p_z(d))}}(x - x_0). \quad (35)$$

Step 4: Now we calculate the state behind the exit surface. To begin with, we have now *two* jump conditions for the momenta at the exit surface

$$
\begin{aligned}
{}^{1}p^{0}_{x1,2}(d',z) &= p_{x1}(d',z) + \frac{m\hbar\Omega_{ba}}{p_{x1}(d',z)}\,(-y(p_{z1}(d',z)) \\
&\quad \pm \sqrt{1 + y^2(p_{z1}(d',z))}\ ), \qquad (36)\\
{}^{2}p^{0}_{x1,2}(d',z) &= p_{x2}(d',z) + \frac{m\hbar\Omega_{ba}}{p_{x2}(d',z)}\,(y(p_{z2}(d',z)) \\
&\quad \pm \sqrt{1 + y^2(p_{z2}(d',z))}\ ). \qquad (37)
\end{aligned}
$$

The components p_{z1} and p_{z2} are continuous at the boundary. The two different ${}^{1,2}p^0_z(d',z)$ lead to two pairs of phases ${}^{1,2}S^0_{1,2}(x,z)$ behind the laser beam.

Without gravity both states ψ_1 and ψ_2 are connected with the same $p_z(d',z)$ so that they give both the *same* two states ψ^0_1 and ψ^0_2. In the case with gravity the situation changes: Owing to the anomalous interaction, we have two different components $p_{z1,2}(d',z)$. The momentum $p_{z1}(d',z)$ belongs to the state ψ_1, and $p_{z,2}(d',z)$ to the state ψ_2. Both states now split into two states: ψ_1 splits into ${}^1\psi^0_1$ and ${}^1\psi^0_2$, and ψ_2 into ${}^2\psi^0_1$ and ${}^2\psi^0_2$. Correspondingly, the phase S_1 splits into ${}^1S^0_{1,2}$ and S_2 into ${}^2S^0_{1,2}$. However, the states ${}^1\psi^0_1$ and ${}^2\psi^0_1$ (and ${}^1\psi^0_2$ and ${}^2\psi^0_2$) are *different*; it is only for vanishing gravity that they coincide and simply add up.

This may be compared with the two different solutions (20) for the z-coordinate. The interaction with the gravitational field changes the velocity and the momentum of the two branches in a different way, thus leading to momenta at the exit surface which are not related by jump conditions corresponding to two energy eigenstates behind the laser beam.

Consequently, for each $\psi_1(d',z)$ and $\psi_2(d',z)$, we have to require the continuity of the wave function separately. Therefore, we get from $\psi_1(d',z)$ the two states

$$
{}^1\psi^0(d',z) = {}^1\alpha^0_1 \begin{pmatrix}1\\0\end{pmatrix} e^{i\,{}^1S^0_1(d',z)} + {}^1\alpha^0_2 \begin{pmatrix}0\\1\end{pmatrix} e^{i\,{}^1S^0_2(d',z)}, \qquad (38)
$$

and from $\psi_2(d',z)$ the two states

$$
{}^2\psi^0(d',z) = {}^2\alpha^0_1 \begin{pmatrix}1\\0\end{pmatrix} e^{i\,{}^2S^0_1(d',z)} + {}^2\alpha^0_2 \begin{pmatrix}0\\1\end{pmatrix} e^{i\,{}^2S^0_2(d',z)}, \qquad (39)
$$

with

$$
{}^1\alpha^0_1 = \alpha_1 \sqrt{A_+(p_{z1}(d',z))}\, e^{-i\,{}^1s'_{11}}, \qquad (40)
$$

$$
\begin{aligned}
{}^{1}\alpha_2^0 &= -\alpha_1 \sqrt{A_-(p_{z1}(d',z))}\, e^{-i\,{}^{1}s'_{21}}, && (41)\\
{}^{2}\alpha_1^0 &= \alpha_2 \sqrt{A_-(p_{z2}(d',z))}\, e^{-i\,{}^{2}s_{12}}, && (42)\\
{}^{2}\alpha_2^0 &= \alpha_2 \sqrt{A_+(p_{z2}(d',z))}\, e^{-i\,{}^{2}s'_{22}}, && (43)
\end{aligned}
$$

with ${}^{i}S_j^0(d',z) - S_i(d',z) =: \hbar\,{}^{i}s'_{ji}$. The total outgoing wave function is

$$
\psi^0(d',z) = {}^{1}\psi^0(d',z) + {}^{2}\psi^0(d',z). \tag{44}
$$

Summary of steps 1 to 4: If we insert the momenta and express them as functions of the initial momentum $p = p(d)$ at the entrance surface, and the $\alpha_{1,2}$ as functions of the $\alpha_{1,2}^0$ of the wave function in front of the laser region, then the total wave function leaving the laser region has the structure

$$
\begin{aligned}
\psi_{\text{out}}(x,z) = {}&{}^{1}\alpha_1^0 \begin{pmatrix}1\\0\end{pmatrix} e^{i\,{}^{1}S_1^0(x,z)} + {}^{1}\alpha_2^0 \begin{pmatrix}0\\1\end{pmatrix} e^{i\,{}^{1}S_2^0(x,z)}\\
&+ {}^{2}\alpha_1^0 \begin{pmatrix}1\\0\end{pmatrix} e^{i\,{}^{2}S_1^0(x,z)} + {}^{2}\alpha_2^0 \begin{pmatrix}0\\1\end{pmatrix} e^{i\,{}^{2}S_2^0(x,z)}
\end{aligned} \tag{45}
$$

for $x > d'$ and where the coefficients ${}^{1,2}\alpha_{1,2}^0$ can be read off from (40)–(43) with $\alpha_{1,2}$ from (29),(30). The first two wave functions are the two energy eigenfunctions belonging to the + branch inside the beam splitter, the last two to the − branch. The first and third wave function describe atoms in the excited state, the second and fourth atoms in the ground state.

From this we can introduce two matrices ${}^{1}\mathbb{S}$ and ${}^{2}\mathbb{S}$ describing the splitting of the two branches

$$
\begin{pmatrix}{}^{1}\alpha_1^0\\ {}^{1}\alpha_2^0\end{pmatrix} = {}^{1}\mathbb{S}\begin{pmatrix}\alpha_1^0\\ \alpha_2^0\end{pmatrix}, \qquad \begin{pmatrix}{}^{2}\alpha_1^0\\ {}^{2}\alpha_2^0\end{pmatrix} = {}^{2}\mathbb{S}\begin{pmatrix}\alpha_1^0\\ \alpha_2^0\end{pmatrix}. \tag{46}
$$

We express the momenta (36),(37) of the outgoing waves with the momenta of the ingoing wave to first order in the gravitational acceleration,

$$
\begin{aligned}
{}^{1}p_{x1,2}^0(d',z) &\approx p_{x1}^0(d,z)\left(1 - \frac{2m\hbar\Omega_{ba}}{(p_{x1}^0(d,z))^2} A_\mp(d)\delta y^+\right), && (47)\\
{}^{2}p_{x1,2}^0(d',z) &\approx p_{x1}^0(d,z)\left(1 - \frac{2m\hbar\Omega_{ba}}{(p_{x1}^0(d,z))^2} A_\mp(d)\delta y^-\right), && (48)
\end{aligned}
$$

with

$$
\delta^\pm y \approx K_g l\left(1 + \frac{\hbar\Omega_{ba}}{2E'}\left(y \pm \sqrt{1+y^2}\right)\right) \tag{49}
$$

$$
\text{with} \quad K_g := \frac{mg}{2\hbar\Omega_{ba}} \frac{\hbar k}{\sqrt{2mE'}}.
$$

Without the gravitational field the outgoing momentum is the same as the incoming one. However, the influence of gravity is modified by the laser beam. For the z component of the momentum we get (see Fig. 3)

$$p^0_{z1,2}(d', z) = p_z(d) + \frac{m^2 g}{\sqrt{2m\mathcal{E}_{1,2}(d)}} l = p^0_z(d) + \frac{m^2 g}{\sqrt{2m\mathcal{E}_{1,2}(d)}} l. \tag{50}$$

Without the anomalous interaction of gravity with the atoms due to the additional periodic potential, both atomic beams will arrive at the exit surface in such a way that for both beams the created beams outside the beam splitter can superpose with the same wavelengths thus giving effectively two outgoing waves. However, due to the anomalous gravitational interaction we get four outgoing waves.

4. OMISSION OF THE GRAVITATIONALLY INDUCED SPLITTING

In an interference experiment with the simple geometry of Fig. 1, these four states propagate to the next beam splitter where they are split into eight beams and a further beam splitter gives 16 beams. The task is to calculate these 16 beams. 8 of them describe atoms in the ground state, the other eight beams atoms in the excited state.

Since this is a huge amount of calculation, we make an approximation which consists of omitting this additional splitting due to the gravitational interaction. That is, we neglect an additional term of the order $\hbar\Omega_{ba}/2E'$ in (49). In this case there will be no additional splitting so that both branches give the same outgoing waves after leaving the laser region.

Consequently, it is no longer necessary to distinguish between the matrices ${}^1\mathbb{S}$ and ${}^2\mathbb{S}$. Instead, the coefficients of the outgoing wave are connected with the coefficients of the ingoing wave by means of

$$\begin{pmatrix} \alpha_1^{0,\text{out}} \\ \alpha_2^{0,\text{out}} \end{pmatrix} = \mathbb{S} \begin{pmatrix} \alpha_1^{0,\text{in}} \\ \alpha_2^{0,\text{in}} \end{pmatrix} \tag{51}$$

with ($A'_\pm := A_\pm(p_z(d'), z)$))

$$\mathbb{S} := \begin{pmatrix} \sqrt{A_+ A'_+}\, e^{i(s_{11} - s'_{11})} + \sqrt{A_- A'_-}\, e^{i(s_{12} - s'_{21})} & \sqrt{A_+ A'_-}\, e^{i(s_{22} - s'_{12})} - \sqrt{A_- A'_+}\, e^{i(s_{21} - s'_{11})} \\ \sqrt{A_- A'_+}\, e^{i(s_{12} - s'_{22})} - \sqrt{A_+ A'_-}\, e^{i(s_{++} - s'_{21})} & \sqrt{A_- A'_-}\, e^{i(s_{21} - s'_{21})} + \sqrt{A_+ A'_+}\, e^{i(s_{22} - s'_{22})} \end{pmatrix} \tag{52}$$

where we approximated $s'_{ij} \approx {}^j s'_{ij}$. This matrix represents the behavior of the beam splitter in the gravitational field. The moduli $|\mathbb{S}_{12}|^2$ and $|\mathbb{S}_{22}|^2$

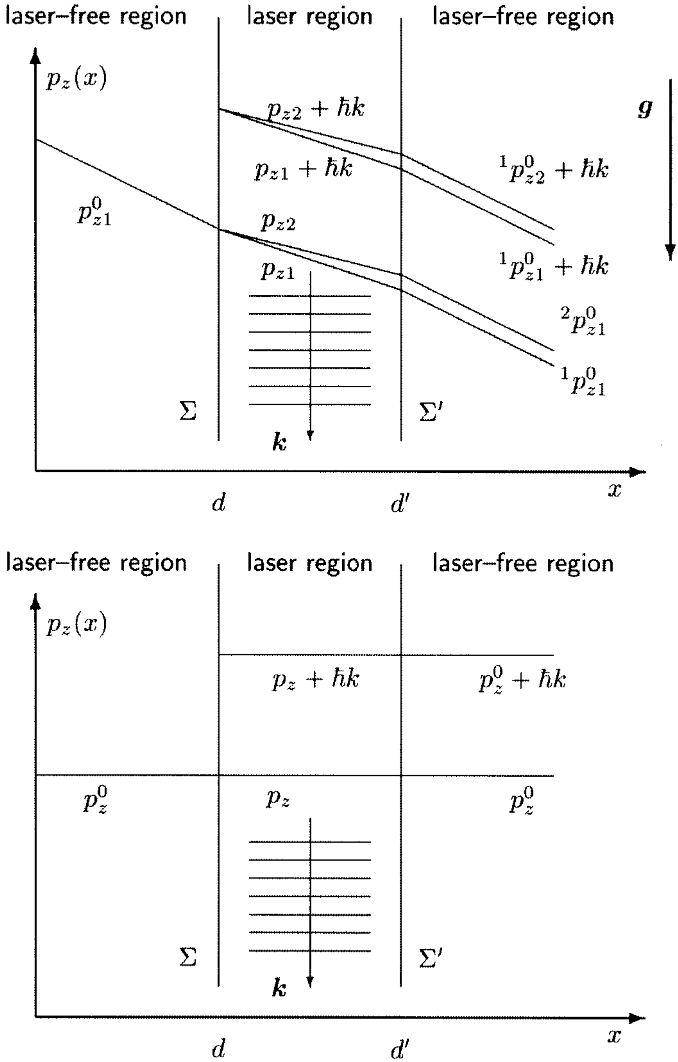

Figure 3. Top: Splitting of the z-component of the momentum in the presence of a gravitational field. The main difference to the case without gravitation is the splitting of p_z inside the laser region (beside the additional stimulation of an additional $p_z + \hbar k$ momentum). An incoming momentum p_{z2} excites the same momenta inside and behind the laser region. Bottom: For comparison: Same process in gravity–free space.

give the probability to find the atom behind the beam splitter in an excited, or ground state, respectively, provided the incoming atom was prepared in the ground state. Similarly, $|\mathbb{S}_{11}|^2$ and $|\mathbb{S}_{12}|^2$ is the probability to find the atom in an excited or ground state, respectively, provided the incoming atom was in an excited state.

With the usual Rabi wavelength $k_R := \beta\sqrt{1+y^2}$, $\beta := m\Omega_{ba}/\sqrt{2mE'}$ (β is a wavelength characteristic for the ratio of Rabi energy and kinetic energy; in other words, it is the spatial version of the Rabi frequency Ω_{ba}: since E' is the kinetic energy of a ground state atom in the laser free region, $E' = \frac{1}{2}mv^2$, we have $\beta = \Omega_{ba}/v$), we define a gravitationally modified Rabi phase

$$S_R^g := k_R l + \frac{y}{\sqrt{1+y^2}}\beta K_g l(d+l) \tag{53}$$

and the corresponding local gravitationally modified Rabi wavelength

$$K_R^g := k_R + \frac{y}{\sqrt{1+y^2}} 2\beta K_g(d+l)\,. \tag{54}$$

The evaluation of the matrix elements gives ($\delta := \beta y$)

$$\mathbb{S}_{11} = e^{-i(\delta+\beta K_g(d+l))l}\left(\cos S_R^g + i\left(\frac{y}{\sqrt{1+y^2}} + \frac{1}{2(1+y^2)^{3/2}}K_g l\right)\sin S_R^g\right), \tag{55}$$

$$\mathbb{S}_{12} = \frac{e^{-i(\delta(2d+l)+\beta K_g l(d+l))}}{\sqrt{1+y^2}}\left(-i\left(1-\frac{y}{2(1+y^2)}K_g l\right)\sin S_R^g - \frac{1}{2\sqrt{1+y^2}}K_g l\cos S_R^g\right), \tag{56}$$

$$\mathbb{S}_{21} = \frac{e^{i(\delta(2d+l)+\beta K_g l(d+l))}}{\sqrt{1+y^2}}\left(-i\left(1-\frac{y}{2(1+y^2)}K_g l\right)\sin S_R^g + \frac{1}{2\sqrt{1+y^2}}K_g l\cos S_R^g\right), \tag{57}$$

$$\mathbb{S}_{22} = e^{i(\delta+\beta K_g(d+l))l}\left(\cos S_R^g - i\left(\frac{y}{\sqrt{1+y^2}} + \frac{1}{2(1+y^2)^{3/2}}K_g l\right)\sin S_R^g\right), \tag{58}$$

where all quantities have to be evaluated at the entrance surface $x = d$. We have used a first order approximation in g and $\hbar\Omega_{ba}/2E'$ in the amplitude as well as in the exponent.

We have always made the calculation to first order in $m\hbar\Omega_{ba}/(p_x^0)^2$.

5. INTERFERENCE EXPERIMENT TAKING INTO ACCOUNT THE MODIFICATION OF BEAM SPLITTING

A sequence of three beam splitters, which realizes the simple interferometer (see Fig. 1), is given by the following procedure: We have at $d = d_1 = 0$ a laser region with length l_1, then at $d = d_2$ a laser region with length l_2 and at $d = d_3$ the third laser region with length l_3. The assemblage of these three laser regions is described by

$$S = \mathbb{S}_3(d_3, l_3)\,\mathbb{S}^0(d_3, d_2)\,\mathbb{S}_2(d_2, l_2)\,\mathbb{S}^0(d_2, d_1)\,;\mathbb{S}_1(d_1, l_1), \tag{59}$$

where $\mathbb{S}^0$ describes the propagation of the atoms in the region between the laser beams. For simplicity, we choose $l_1 = l_2 = l_3 = l$ and we choose l in such a way that at the first beam splitter is exactly a $\pi/2$ beam in the absence of gravitation. The condition for that is $k_R l = \pi/4$. The second beam splitter is assumed to have the length $2l$ and the third has the length l. The distance between the laser regions is d.

If we neglect terms quadratic in the gravitational acceleration in the amplitude and in the phase separately, then we get as observed intensity of the atoms leaving the interferometer in the excited state

$$I_2 = \frac{1}{2}\left[1 - \cos\left(\frac{mgk}{E}(2d^2 + 2dl - l^2)\right)\right.$$
$$\left. + K_g\left(d + \frac{3}{2}l\right)\sin\left(\frac{mgk}{E}(2d^2 + 2dl - l^2)\right)\right], \tag{60}$$

provided that all incoming atoms are in the ground state. In the case $l \longrightarrow 0$ and $K_g \longrightarrow 0$ we get for this phase shift the classical result for infinitely thin laser beams,

$$\delta\phi = kg\frac{d^2}{v^2}, \tag{61}$$

which corresponds directly to the phase shift for the time pulsed interferometers $\delta\phi = kgT^2$, where T is the time between the laser pulses. Note that because of $m/F_2 \sim 1/N^2$ for (61) and also for the more general case (60) there appears no mass in the phase shift. Consequently, the equivalence principle is fulfilled — see also [15–17].

There are two causes for a modification of the phase shift (61): First, the effect that even if at the first beam splitter the atoms are in resonance

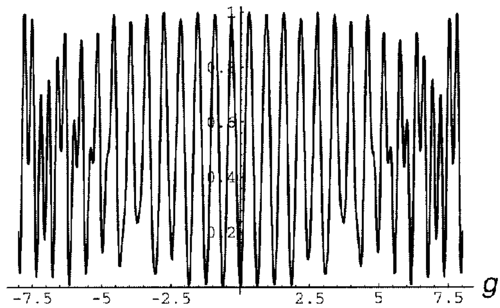

Fig. 4(a).

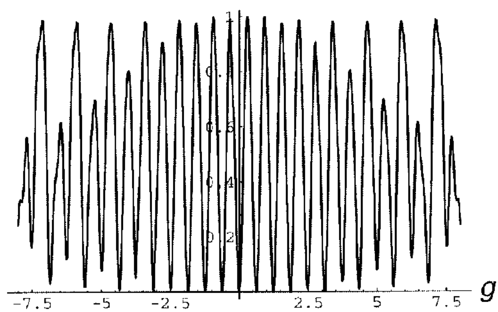

Fig. 4(b).

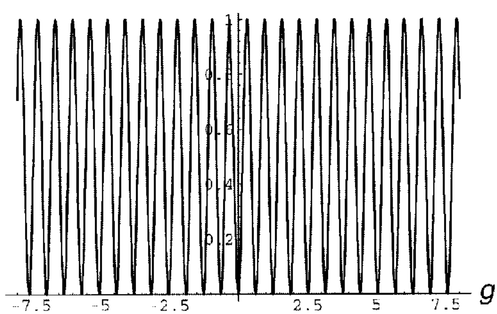

Fig. 4(c).

Figure 4. Upper: Typical interference pattern for an interference experiment in the gravitational field taking into account the modification of the beam splitting process induced by the gravitational field. Lower Left: This figure is for vanishing width of the beam splitter $l = 0$. Lower Right: For comparison: Interference pattern for the same specifications as in upper figure but neglecting all gravitational modifications of the beam splitting process.

with the laser, the additional velocity gained during the flight to the second beam splitter brings the atoms slightly out of resonance. Second, the modified acceleration inside the beam splitter. The first modification vanishes for $K_g \to 0$, the second for $l \to 0$.

Finally we present some diagrams of possible interference patterns. In addition, we specialized to the case that the atomic beam is at resonance $y = 0$ at the entrance surface of the first beam splitter. We show the possible modifications for variations of the thickness of the laser regions.

The corresponding diagrams for neutron interferometry in a gravitational field have been presented in [15]. A characteristic feature of their result was that the contrast decreases for larger accelerations. This is due to the fact that in neutron diffraction there is absorption of the neutrons if they do not hit the crystal surface with the Bragg angle. The presence of a gravitational acceleration acts in this way.

In the case of atoms interacting with a laser beam no such absorption will occur. Therefore we do not expect to lose the contrast by varying the gravitational acceleration. A specific feature of our diagrams is obviously the superposition of two waves. This can be understood from the structure of the matrix elements describing the beam splitter. This matrix contains two wavelengths: first the local Rabi wavelength K_R^g and second the wavelength δ in the phase of each matrix element. These two wavelengths give the beat structure of the first two patterns in Fig. 4. For comparison, the lower left diagram of Fig. 4 is the interference pattern without gravitational modification of the beam splitting process. It is calculated by neglecting in the matrix elements all modifications of the amplitudes and of the beam splitting process, that is, by setting $K_g = 0$.

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