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Rabi oscillations in gravitational fields: Exact solution

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Abstract

The exact solution for a two-level atom interacting with a laser field in a gravitational field is given. The modifications due to a weak gravitational field are discussed and some applications are proposed.

1. Introduction

Since any quantum optical experiment in the laboratory is actually made in a noninertial frame it is important to estimate the influence of the earth's acceleration on the outcome of these experiments. Here we treat the influence of acceleration on a two-level atom interacting with a monochromatic laser field. Thereby we present the exact solution to this problem and we discuss some special cases and implications.

We show how the Rabi oscillations which usually depend on the interaction energy, the detuning, the wave vector of the laser beam and on the momentum of the atom, change in the presence of acceleration. For practical purposes these concepts are essential for an exact description of atomic beam interferometry [1] and any effects of two-level systems travelling through strong laser fields [2] in the presence of gravitational fields. Our formalism should be also applicable to atoms submitted to laser pulses in optomagnetic [3,4] or gravitational traps [5].

Our result describes the evolution of a two-level system interacting with a laser beam in the presence of a gravitational field. The Rabi oscillations as well as the population transfer will be modified by the gravitational field. The oscillations increase in time and the population transfer decreases. To some extend the present effect is similar to the effect of a frequency chirped laser interacting with a two-level atom, discussed by Horwitz [6], and to the motion of an atom in a laser beam with curved wave front [7].

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2. Basic equations

We start with the Schrödinger equation for a two-level atom,

$$i\hbar\frac{\partial\Psi(\boldsymbol{r},t)}{\partial t} = \left(-\frac{\hbar^2}{2m}\Delta + H_0 + H_d(\boldsymbol{r},t) + H_g(\boldsymbol{r})\right)\Psi(\boldsymbol{r},t). \tag{1}$$

With the rotating wave approximation

$$H_0 = \begin{pmatrix} E_b & 0 \\ 0 & E_a \end{pmatrix} = \tfrac{1}{2}(E_a + E_b) + \tfrac{1}{2}\hbar\omega_{ba}\sigma_3, \qquad \text{with } \omega_{ba} := \frac{1}{\hbar}(E_b - E_a), \tag{2}$$

$$H_d(\boldsymbol{r},t) = -\hbar\Omega_{ba}\begin{pmatrix} 0 & e^{-i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r}+\phi)} \\ e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r}+\phi)} & 0 \end{pmatrix}, \tag{3}$$

$$H_g(\boldsymbol{r}) = -m\boldsymbol{g}\cdot\boldsymbol{r}, \tag{4}$$

where E_b and E_a are the upper and lower energy level of the atom, m is the mass of the atom and $\boldsymbol{g}$ is the earth's acceleration. $\Omega_{ba} = (e/2\hbar)|\langle a|dE|b\rangle|$ is the Rabi frequency whereby for simplicity we assumed nondegenerate levels a and b and considered only a scalar electric dipole interaction. ω, $\boldsymbol{k}$ and ϕ are the frequency, the wave vector, and a constant phase of the laser beam. We assume that the energy levels E_a and E_b are not modified by the gravitational field or during acceleration. We neglect any relaxation effects like finite lifetimes and dephasing. This two-level description of our atoms can also be applied to three-level systems excited by two laser fields in the case of an intermediate nonresonant state out of resonance, e.g. Raman two-photon excitations, or to two-level systems interacting with a standing wave, because in these cases one can write an effective Hamiltonian which has the above form.

In order to remove the time-dependence $\omega t + \phi$ in the dipole interaction term, we make the transformation $\psi(\boldsymbol{r},t) := e^{-i(\omega t+\phi)\sigma_3/2}\,\Psi(\boldsymbol{r},t)$ and get

$$i\hbar\partial_t\psi(\boldsymbol{r},t) = [H(\boldsymbol{r},\widehat{p}) - \tfrac{1}{2}\hbar\omega\sigma_3 + H_a + H_d'(\boldsymbol{r})]\psi(\boldsymbol{r},t), \tag{5}$$

with

$$H_d'(\boldsymbol{r},t) = H_d'(\boldsymbol{r}) = -\hbar\Omega_{ba}\begin{pmatrix} 0 & e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \\ e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} & 0 \end{pmatrix}. \tag{6}$$

3. The exact solution

We write (5) explicitly in two-component notation

$$i\hbar\partial_t\psi_b = -\frac{\hbar^2}{2m}\Delta\psi_b + E_b\psi_b - m\boldsymbol{g}\cdot\boldsymbol{r}\psi_b - \hbar\Omega_{ba}\,e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\,\psi_a - \tfrac{1}{2}\hbar\omega\psi_b, \tag{7a}$$

$$i\hbar\partial_t\psi_a = -\frac{\hbar^2}{2m}\Delta\psi_a + E_a\psi_a - m\boldsymbol{g}\cdot\boldsymbol{r}\psi_a - \hbar\Omega_{ba}\,e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\,\psi_b + \tfrac{1}{2}\hbar\omega\psi_a, \tag{7b}$$

apply a modified Fourier transformation which includes the wave vector k,

$$\psi_b(\boldsymbol{r},t) = \frac{1}{(2\pi)^{3/2}}\int e^{i(\boldsymbol{K}+\boldsymbol{k})\cdot\boldsymbol{r}}\,a_b(\boldsymbol{K},t)\,d^3K, \tag{8a}$$

$$\psi_a(\boldsymbol{r},t) = \frac{1}{(2\pi)^{3/2}}\int e^{i\boldsymbol{K}\cdot\boldsymbol{r}}\,a_a(\boldsymbol{K},t)\,d^3K, \tag{8b}$$

and get two partial differential equations in momentum space,

$$i\hbar\partial_t a_b = \frac{\hbar^2(\boldsymbol{K}+\boldsymbol{k})^2}{2m}a_b + (E_b - \tfrac{1}{2}\hbar\omega)a_b - \mathrm{i}m\boldsymbol{g}\cdot\nabla_K a_b - \hbar\Omega_{ba}a_a, \tag{9a}$$

$$i\hbar\partial_t a_a = \frac{\hbar^2 K^2}{2m}a_a + (E_a + \tfrac{1}{2}\hbar\omega)a_a - \mathrm{i}m\boldsymbol{g}\cdot\nabla_K a_a - \hbar\Omega_{ba}a_b. \tag{9b}$$

The next step is to make the transformation

$$a(\boldsymbol{K},t) = \exp\left[-\frac{1}{2}\frac{\mathrm{i}}{\hbar}\left(\frac{\hbar^2(\boldsymbol{K}+\boldsymbol{k})^2}{2m} + \frac{\hbar^2 K^2}{2m} + E_a + E_b\right)t\right]\tilde{a}(\boldsymbol{K},t), \tag{10}$$

which removes the mean free energy. The additional transformation

$$\tilde{a}(\boldsymbol{K},t) = \exp\{\mathrm{i}[\tfrac{1}{2}(\boldsymbol{K}+\tfrac{1}{2}\boldsymbol{k})\cdot\boldsymbol{g}t^2 - (m/6\hbar)g^2t^3]\}\hat{a}(\boldsymbol{K},t) \tag{11}$$

removes parts of the acceleration terms. We end up with the coupled equations

$$\left(\partial_t + \frac{m\boldsymbol{g}}{\hbar}\cdot\nabla_K\right)\hat{a}_b = -\mathrm{i}\Omega_{ba}y(\boldsymbol{K})\hat{a}_b + \mathrm{i}\Omega_{ba}\hat{a}_a, \tag{12a}$$

$$\left(\partial_t + \frac{m\boldsymbol{g}}{\hbar}\cdot\nabla_K\right)\hat{a}_a = \mathrm{i}\Omega_{ba}y(\boldsymbol{K})\hat{a}_a + \mathrm{i}\Omega_{ba}\hat{a}_b, \tag{12b}$$

with $y(\boldsymbol{K}) = (1/2\Omega_{ba})[(\hbar\boldsymbol{k}/2m)\cdot(2\boldsymbol{K}+\boldsymbol{k}) - \Delta]$ where $\Delta = \omega - \omega_{ba}$ is the detuning. The above equations have the form of a vector-valued evolution equation,

$$M^\mu\partial_\mu\hat{a}(\boldsymbol{K},t) = \mathrm{i}\Omega_{ba}\begin{pmatrix} -y(\boldsymbol{K}) & 1 \\ 1 & y(\boldsymbol{K})\end{pmatrix}\hat{a}(\boldsymbol{K},t), \tag{13}$$

with $M^0 = 1, M^i \leftrightarrow \boldsymbol{M} = (m\boldsymbol{g}/\hbar), \partial_0 = \partial_t, \partial_i \leftrightarrow \nabla_K$. From M^μ we can define a curve $C^\mu(\tau) = \binom{\tau}{\boldsymbol{K}+m\boldsymbol{g}/\hbar\tau}$ so that $M^\mu = (\mathrm{d}/\mathrm{d}\tau)C^\mu(\tau)$. Defining $\hat{a}(\tau) := \hat{a}(\boldsymbol{K}+(m\boldsymbol{g}/\hbar)\tau,\tau)$ we have a system of ordinary differential equations,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\hat{a}(\tau) = \mathrm{i}\Omega_{ba}\begin{pmatrix} -y(\boldsymbol{K}) - \frac{1}{2}\boldsymbol{g}\cdot\boldsymbol{k}\tau & 1 \\ 1 & y(\boldsymbol{K}) + \frac{1}{2}\boldsymbol{g}\cdot\boldsymbol{k}\tau\end{pmatrix}\hat{a}(\tau). \tag{14}$$

Introducing a new parameter $\lambda := \mathrm{e}^{\mathrm{i}\pi/4}[(2\Omega_{ba}y(\boldsymbol{K})/\sqrt{kg}) + \sqrt{gk}\tau]$, and $\check{a}_b(\lambda) := -\mathrm{e}^{\mathrm{i}\pi/4}(\Omega_{ba}/\sqrt{kg})\hat{a}_b(\lambda)$ (for notational simplicity we write kg instead of $\boldsymbol{k}\cdot\boldsymbol{g}$) this gives

$$\frac{\mathrm{d}\check{a}_b(\lambda)}{\mathrm{d}\lambda} + \tfrac{1}{2}\lambda\check{a}_b(\lambda) + \mathrm{i}\frac{\Omega_{ba}^2}{kg}\hat{a}_a(\lambda) = 0, \tag{15a}$$

$$\frac{\mathrm{d}\hat{a}_a(\lambda)}{\mathrm{d}\lambda} - \tfrac{1}{2}\lambda\hat{a}_a(\lambda) + \check{a}_b(\lambda) = 0. \tag{15b}$$

Comparison with the recursion relation of the parabolic cylinder functions [8] shows that

$$\check{a}_{b1}(\lambda) = D_{p+1}(\lambda)b_{01}(\boldsymbol{K}), \tag{16a}$$

$$\check{a}_{b2}(\lambda) = D_{-p-2}(-\mathrm{i}\lambda)b_{02}(\boldsymbol{K}), \tag{16b}$$

$$\hat{a}_{a1}(\lambda) = D_p(\lambda)b_{01}(\boldsymbol{K}), \tag{16c}$$

$$\hat{a}_{a2}(\lambda) = D_{-p-1}(-\mathrm{i}\lambda)b_{02}(\boldsymbol{K}), \tag{16d}$$

where $p+1=-\mathrm{i}(\Omega_{ba}^2/kg)$ are the solutions of (15a), (15b). The initial state is represented by $b_{01,2}(\boldsymbol{K})$. Tracing the definitions back the solution is then given by

$$\psi(\boldsymbol{r},t)=\frac{1}{(2\pi)^{3/2}}\int \mathrm{e}^{\mathrm{i}[K+(mg/\hbar)t]x}\,\mathrm{e}^{-(\mathrm{i}/\hbar)E_g(t)} \times\begin{pmatrix}\mathrm{e}^{\mathrm{i}\pi/4}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}[-(\sqrt{kg}/\Omega_{ba})D_{-\nu}(\lambda)b_{01}(\boldsymbol{K})+(\Omega_{ba}/\sqrt{kg})D_{\nu-1}(-\mathrm{i}\lambda)b_{02}(\boldsymbol{K})]\\ D_{-\nu-1}(\lambda)b_{01}(\boldsymbol{K})+D_{\nu}(-\mathrm{i}\lambda)b_{02}(\boldsymbol{K})\end{pmatrix}\mathrm{d}^3K, \tag{17}$$

with

$$E_g(t)=\frac{\hbar^2K^2}{2m}t+\tfrac{1}{2}\hbar(\boldsymbol{K}+\tfrac{1}{2}\boldsymbol{k})\cdot\boldsymbol{g}t^2+\tfrac{1}{6}mg^2t^3+\hbar\Omega_{ba}y(\boldsymbol{K})t+(E_a+\tfrac{1}{2}\hbar\omega)t, \tag{18}$$

$$\nu=\mathrm{i}\frac{\Omega_{ba}^2}{kg}. \tag{19}$$

The functions $b_{01,2}(\boldsymbol{K})$ can be determined by the initial value of the wave function. We finally end up with the general solution

$$\psi(\boldsymbol{r},t)=\frac{1}{(2\pi)^{3/2}}\int \mathrm{e}^{\mathrm{i}[K+(mg/\hbar)t]\cdot r}\,\mathrm{e}^{-(\mathrm{i}/\hbar)E_g(t)}\,\mathrm{e}^{\mathrm{i}\pi\nu/2}\begin{pmatrix}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}a_{b0}(\boldsymbol{K})\\ a_{a0}(\boldsymbol{K})\end{pmatrix}\mathrm{d}^3K, \tag{20}$$

with

$$A:=\frac{\Omega_{ba}^2}{kg}D_{\nu-1}(-\mathrm{i}\lambda)D_{-\nu-1}(\lambda_0)+D_{-\nu}(\lambda)D_{\nu}(-\mathrm{i}\lambda_0), \tag{21a}$$

$$B:=\mathrm{e}^{-\mathrm{i}\pi/4}\frac{\Omega_{ba}}{\sqrt{kg}}[D_{\nu-1}(-\mathrm{i}\lambda)D_{-\nu}(\lambda_0)-D_{-\nu}(\lambda)D_{\nu-1}(-\mathrm{i}\lambda_0)], \tag{21b}$$

$$C:=\mathrm{e}^{\mathrm{i}\pi/4}\frac{\Omega_{ba}}{\sqrt{kg}}[D_{\nu}(-\mathrm{i}\lambda)D_{-\nu-1}(\lambda_0)-D_{-\nu-1}(\lambda)D_{\nu}(-\mathrm{i}\lambda_0)], \tag{21c}$$

$$D:=\frac{\Omega_{ba}^2}{kg}D_{-\nu-1}(\lambda)D_{\nu-1}(-\mathrm{i}\lambda_0)+D_{\nu}(-\mathrm{i}\lambda)D_{-\nu}(\lambda_0) \tag{21d}$$

($\lambda_0=2\,\mathrm{e}^{\mathrm{i}\pi/4}(\Omega_{ba}/\sqrt{kg})y(\boldsymbol{K})$). For $t\to 0$ we recover the identity transformation. The off-diagonal elements are scaled with $\Omega_{ba}/\sqrt{kg}$. This means that for $\Omega_{ba}\to 0$ (vanishing influence of the laser beam) or $kg\to\infty$ (strong gravitational field) the Rabi oscillations die out.

The variable λ can be rewritten as

$$\lambda=\mathrm{e}^{\mathrm{i}\pi/4}\left(2\frac{\Omega_{ba}}{\sqrt{kg}}y(\boldsymbol{K})+\sqrt{kg}t\right)=2\,\mathrm{e}^{\mathrm{i}\pi/4}\frac{\Omega_{ba}}{\sqrt{kg}}y(\boldsymbol{K}+(m\boldsymbol{g}/\hbar)t). \tag{22}$$

Therefore λ essentially represents the parameter $y(\boldsymbol{K})$ with time-dependent momentum $\hbar\boldsymbol{K}$ changing according to the classical law. In addition, $\lambda=0$, or equivalently,

$$\boldsymbol{k}\cdot\left(\boldsymbol{g}t+\frac{\hbar\boldsymbol{K}}{m}\right)+\frac{\hbar k^2}{2m}-\Delta=0 \tag{23}$$

can be given a physical interpretation: (23) is fulfilled for that moment t, when the gravitational induced Doppler shift $\boldsymbol{k}\cdot\boldsymbol{g}t$ cancels the Doppler shift from the initial velocity $v_0=\hbar\boldsymbol{K}/m$, the recoil shift $\hbar k^2/2m$, and the detuning Δ. From the viewpoint of the Rabi oscillations this is the "turning point" of the motion of the atom in the gravitational field. (The time t_p of the classical turning point is given by (for an initial momentum

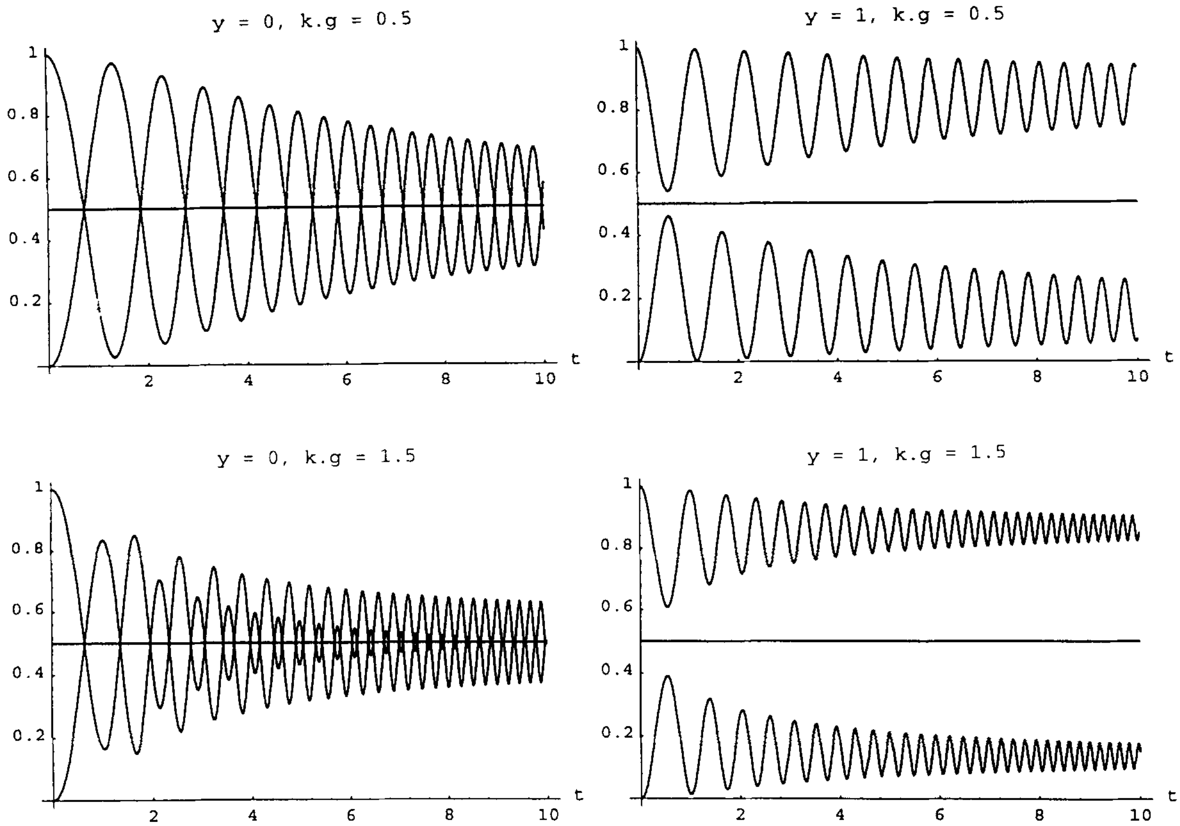

Fig. 1. Rabi oscillations for different values of y and $\boldsymbol{k}\cdot\boldsymbol{g}$ (for $\Omega_{ba} = 1$).

$\boldsymbol{K}$ directing antiparallel to the acceleration $\boldsymbol{g}$) $\hbar K/m = gt_p$.) At that moment the Rabi frequency is minimal and the population inversion maximal. After and before that moment the Rabi frequency increases and the amplitudes $|\psi_a|^2$ and $|\psi_b|^2$ decrease.

Fig. 1 presents features of the solution for an initial state $a_a(\boldsymbol{K},0) = 0$, $a_b(\boldsymbol{K},0) = \delta^3(\boldsymbol{K} - \boldsymbol{K}_0)$, that is, for an initially fully occupied upper level with initial momentum $\boldsymbol{K}_0$. The upper curve (starting at 1) is $|A(t)|^2$ and describes the occupation of the level b, the lower curve (starting at 0) is $|C(t)|^2$ and shows the occupation of the lower level. The two diagrams on the left are for $y = 0$ and $g = 0.5$ and 1.5; the two diagrams on the right are for the same g-values but for $y = 1$. The population transfer is best for $y = 0$ and $t = 0$ which represents the "turning point" of the evolution of the two-level system. The amplitudes of the oscillations are smaller for larger values of y and decrease more rapidly for larger accelerations. For increasing time the oscillations decrease and freeze at a specific value (see below). The instantaneous frequency of the oscillations increases with time.

If one assumes an initial plane wave with momentum K_0, that is, $b_{01,2}(\boldsymbol{K}) = \alpha_{1,2}\delta^3(\boldsymbol{K} - \boldsymbol{K}_0)$, then the resulting wave functions can be easily obtained by integration. Operating with the momentum operator on these functions, we get

$$p\psi_b(\boldsymbol{r},t) = \left(\boldsymbol{K}_0 + \boldsymbol{k} + \frac{m\boldsymbol{g}}{\hbar}t\right)\psi_b(x,t), \qquad p\psi_a(x,t) = \left(\boldsymbol{K}_0 + \frac{m\boldsymbol{g}}{\hbar}t\right)\psi_a(x,t). \tag{24}$$

This means that the momentum of plane waves changes with the acceleration according to the classical law.

The above solution (20) is also valid if we replace Ω_{ba} by some step function $\Omega_{ba}(t) = \Omega_{ba}^0\theta(t-t_1)$ or by a pulse with rectangular shape $\Omega_{ba}(t) = \Omega_{ba}^0(\theta(t-t_1) - \theta(t-t_2))$ for $t_2 > t_1$. This can be proven to be very general: We insert into our solution (20), which is differentiable in Ω_{ba}, a time-dependent $\Omega_{ba}(t)$ defining a function $\psi_{\Omega(t)}(t,x)$. This is the usual way to an adiabatic treatment of time-dependent effects. The time evolution of the fields $\psi_{\Omega(t)}(t,x)$ for a short time interval t, $t+\delta t$ is given by

$$\psi_{\Omega(t)}(t+\delta t,x) = \psi_{\Omega(t)}(t,x) + \frac{\partial}{\partial t}\psi_{\Omega(t)}(t,x)\delta t + \frac{\partial}{\partial\Omega}\psi_{\Omega(t)}(t,x)\frac{\partial\Omega(t)}{\partial t}\delta t, \tag{25}$$

$\psi_{\Omega(t)}(t,x)$ is still a solution of the original equation with Ω replaced by $\Omega(t)$ if the second term vanishes which is the case if $\Omega(t) \sim \theta(t)$. Therefore, our solution (20) is valid also for rectangular time-profiles. For all other cases, for example for Gaussian time profiles of the laser pulse, this is not true.

4. Discussion

4.1. Strong gravitational field or vanishing laser beam

Now we treat $g \to \infty$, or more exactly, $gk \to \infty$ which gives the same results as $\Omega_{ba} \to 0$. In these cases we have to consider $D_{\pm\nu}(z)$ and $D_{\pm\nu-1}(z)$ for $\nu \to 0$ and any value of z. We get

$$A = \frac{\Omega_{ba}^2}{kg}D_{-1}(-\mathrm{i}\lambda)D_{-1}(\lambda_0) + D_0(\lambda)D_0(\lambda_0) \to D_0(\lambda)D_0(\lambda_0), \tag{26a}$$

$$B = \mathrm{e}^{-\mathrm{i}\pi/4}\frac{\Omega_{ba}}{\sqrt{kg}}[D_{-1}(-\mathrm{i}\lambda)D_0(\lambda_0) - D_0(\lambda)D_{-1}(\lambda_0)] \to 0, \tag{26b}$$

$$C = \mathrm{e}^{\mathrm{i}\pi/4}\frac{\Omega_{ba}}{\sqrt{kg}}[-D_{-1}(\lambda)D_0(\lambda_0) + D_0(-\mathrm{i}\lambda)D_{-1}(\lambda_0)] \to 0, \tag{26c}$$

$$D = \frac{\Omega_{ba}^2}{kg}D_{-1}(\lambda)D_{-1}(\lambda_0) + D_0(-\mathrm{i}\lambda_1)D_0(\lambda_0) \to D_0(-\mathrm{i}\lambda)D_0(\lambda_0). \tag{26d}$$

With the explicit representation of $D_n(z)$ for integer n (see Ref. [8]) we arrive with (18) at

$$\psi(x,t) \to \frac{1}{(2\pi)^{3/2}} \times \int \begin{pmatrix} \mathrm{e}^{\mathrm{i}[K+k+(mg/\hbar)t]\cdot r} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}[K+(mg/\hbar)t]\cdot r} \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-(\mathrm{i}/\hbar)E_b(K+k,t)} & 0 \\ 0 & \mathrm{e}^{-(\mathrm{i}/\hbar)E_a(K,t)} \end{pmatrix} \begin{pmatrix} a_{b0}(K) \\ a_{a0}(K) \end{pmatrix} \mathrm{d}^3K, \tag{27}$$

with $E_{b,a}(K,t) = (\hbar^2/2m)K^2 + \frac{1}{2}\hbar(K+k)\cdot gt^2 + \frac{1}{6}mg^2t^3 + (E_{b,a} \mp \frac{1}{2}\hbar\omega)t$. This describes the free fall of a two-level system in a uniform gravitational field.

4.2. Long-time asymptotics

We consider $t \to \infty$, that is, $|\lambda| \to \infty$. For the moment we restrict to Ω_{ba}, y, and $kg > 0$ so that $\arg(\lambda) = \frac{1}{4}\pi$ and $\arg(-\mathrm{i}\lambda) = -\frac{1}{4}\pi$. We can apply [8]

$$D_p(z) \approx \mathrm{e}^{-z^2/4}z^p\left(1 - \frac{p(p-1)}{2z^2} + \mathrm{O}(z^{-4})\right). \tag{28}$$

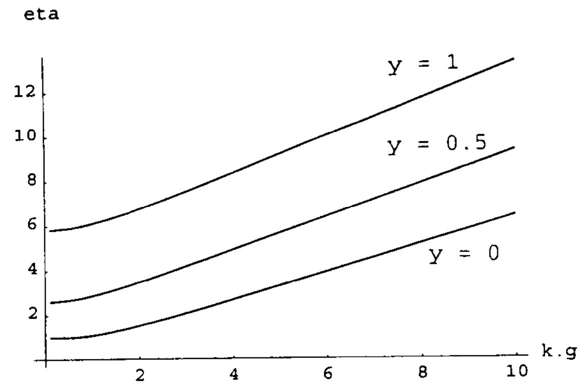

Fig. 2. The ratio η as function of $\boldsymbol{k}\cdot\boldsymbol{g}$ for different values of y (for $\Omega_{ba}=1$).

We take the limits of the modulus of A and C for large t which remains constant: the population of the two-level system freezes with increasing time. If for $t=0$ the level b was fully populated and level a was empty, we have for the ratio η of the population for $t\to\infty$

$$\eta(kg)=\lim_{t\to\infty}\frac{|A|^2}{|C|^2}=\frac{\sqrt{kg}}{\Omega_{ba}}\frac{|D_\nu(-\mathrm{i}\lambda_0)|^2}{|D_{-\nu-1}(\lambda_0)|^2}. \tag{29}$$

This asymptotic expansion is valid for $|z|\gg|p|$, that is, for $t\gg[\Omega_{ba}^2/(kg)^{3/2}]$ and $t\gg 1/\sqrt{kg}$. For $\Omega_{ba}\sim 10^6$ s^{-1} and $kg\sim 10^6\ \mathrm{s}^{-2}$ this means $t\gg 10^3$ s. The ratio grows for increasing values of kg as can be seen in Fig. 2 where we have shown $\eta(kg)$ for different values of y.

4.3. Weak gravitational field

We consider now the case of a weak gravitational field $\boldsymbol{g}\to 0$ whereby the last condition can be replaced by $kg\to 0$ because only this combination is relevant for the two-level system. We get the first modification of the Rabi oscillations induced by the gravitational acceleration $\boldsymbol{g}$.

In this case the modulus of the argument as well as of the parameter of the parabolic cylinder functions tend to infinity: We need an expansion of $D_p(z)$ for the case $|z|\to\infty$ and $|p|\to\infty$. Such an expansion for complex parameters and complex variables has been given in Ref. [9]. Using this result we have for $|\lambda|\to\infty$ and $|p|\to\infty$

$$D_p(\lambda)\sim g_p\frac{1}{(\kappa^2-1)^{1/4}}\,\mathrm{e}^{-\mu^2\xi}\left(1+\frac{1}{\mu}\frac{\kappa^3-6\kappa}{24(\kappa^2-2)^{3/2}}+\ldots\right), \tag{30}$$

with

$$\mu^2=2p+1, \tag{31}$$

$$\kappa=\frac{\lambda}{\sqrt{2+4p}}, \tag{32}$$

$$\xi=-\tfrac{1}{2}\kappa\sqrt{\kappa^2-1}-\tfrac{1}{2}\ln(\kappa+\sqrt{\kappa^2-1}), \tag{33}$$

$$g_p=2^{-\mu^2/4-1/4}\mathrm{e}^{-\mu^2/4}\,\mu^{\mu^2/2-1/2}(1-1/24\mu+\ldots). \tag{34}$$

The advantage of this kind of approximation is that the amplitude and the phase represented by ξ are treated separately. During the lengthy calculations one has to take care of the various branch cuts by taking the square root and the logarithm. The phase gives to first order the gravitationally modified Rabi phase

$$\Omega_{\rm R}(t) = \Omega_{ba}\sqrt{1+y^2}\,t + \frac{1}{4}\frac{y}{\sqrt{1+y^2}}kgt^2. \qquad (35)$$

The corresponding modified instantaneous Rabi frequency $\Omega = (\mathrm{d}/\mathrm{d}t)\Omega_{\rm R}(t) = \Omega_{ba}\sqrt{1+y^2} + \frac{1}{2}(y/\sqrt{1+y^2}) \times kgt$ increases with time. This effect can also be seen in Fig. 1. The relative correction of the usual Rabi frequency is $\delta\Omega = \frac{1}{2}[y/(1+y^2)](kg/\Omega_{ba})t$. For typical values $k \sim 10^6$ m^{-1}, $\Omega_{ba} \sim 10^7$ s^{-1}, $y \sim 1$, and an interaction time of the atom with the laser beam within an atom beam interferometer $t \sim 10^{-3}$ s we have $\delta\Omega \sim 10^{-3}$. This is well within the accuracy of present-day apparatus. It is remarkable that one gets the same result by expanding the usual formula for the Rabi frequency in the case that one treats the atomic motion classically and the interaction in the atomic rest frame with a transformed phase for the electric field.

Taking the amplitude into account, we get as result for the various parts of the evolution matrix

$$A = \cos\Omega_{\rm R}(t) - \mathrm{i}\left(\frac{y}{\sqrt{1+y^2}} - \alpha(t)\frac{y}{(1+y^2)^2}\right)\sin\Omega_{\rm R}(t), \qquad (36a)$$

$$B = \mathrm{i}\left(\frac{1}{\sqrt{1+y^2}} + \alpha(t)\frac{y^2}{(1+y^2)^2}\right)\sin\Omega_{\rm R}(t) - \frac{\alpha(t)}{1+y^2}\cos\Omega_{\rm R}(t) + \frac{\alpha(t)}{4}\frac{y^2}{(1+y^2)^{3/2}}\sin\Omega_{\rm R}(t), \qquad (36b)$$

$$C = \mathrm{i}\left(\frac{1}{\sqrt{1+y^2}} + \alpha(t)\frac{y^2}{(1+y^2)^2}\right)\sin\Omega_{\rm R}(t) + \frac{\alpha(t)}{1+y^2}\cos\Omega_{\rm R}(t) - \frac{\alpha(t)}{4}\frac{y^2}{(1+y^2)^{3/2}}\sin\Omega_{\rm R}(t), \qquad (36c)$$

$$D = \cos\Omega_{\rm R}(t) + \mathrm{i}\left(\frac{y}{\sqrt{1+y^2}} - \alpha(t)\frac{y}{(1+y^2)^2}\right)\sin\Omega_{\rm R}(t), \qquad (36d)$$

with $\alpha(t) = (kg/4\Omega_{ba})t$. For $kg \to 0$ we recover the exact result for the Rabi oscillations in gravity-free space.

5. Outlook

Starting from our exact solution some phenomena may be treated theoretically in future. For example the effect of the laser beam splitters in atom beam interferometry may be treated in an exact manner for square pulses [10,11]. Since atom beam interferometry is very sensitive to the influence of acceleration it is necessary to have exact results for the beam splitters in gravitational fields in order to be able to interpret the results of such experiments in a proper way. The results of the present Letter can also be applied to an exact description of dynamical neutron diffraction in the presence of gravitational fields, which has importance for the interpretation of the COW experiment [12,13].

Also the abnormal behaviour of matter waves in periodic external potentials which has been explored for neutrons in crystal lattices by Zeilinger et al. [14] leading e.g. to negative effective mass tensors, can now be treated in an exact manner. It is of course not necessary to use gravity as external force. Electromagnetic forces acting upon charged or neutral objects (atoms, molecules) with electric or magnetic dipole can be used as well.

Also Marzlin and Audretsch [15] (see also Ref. [16]) treated the same problem, but in an operator formalism, and arrived at similar results.

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