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Amplification of atomic fields by stimulated emission of atoms ★

Ch.J. Bordé

Laboratoire de Physique des Lasers, URA du CNRS 282, Université Paris-Nord, Villetaneuse, France
Laboratoire de Gravitation et Cosmologie Relativistes, URA du CNRS 769, Université Pierre et Marie Curie, Paris, France

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Abstract

We discuss the possibility of amplifying matter-waves by stimulated emission of massive bosons. Specifically, we consider the induced dissociation of molecules under the influence of an incident atomic wave and compare this to the special case of the induced emission of ground state atoms from excited atoms. General formulas for the gain and for the spontaneous dissociation rate, which apply to both cases, are derived from relativistic quantum field theory.

1. Introduction

The recent development of atom optics and interferometry and especially of atomic resonators has demonstrated that one could often exchange the roles of atoms and photons in optical devices in order to design an atomic wave device. This is part of a general view of the world of photons and atoms as elementary field particles which can mutually scatter each other in a reciprocal way. For example, one could imagine to achieve a device equivalent to optical lasers where photons are replaced by integer spin atoms and where the atoms are reflected or trapped by light ("atom laser", "atomaser" or "ataser").

One way towards this goal, which is currently being explored, is to achieve Bose–Einstein condensation with a collection of cold atoms in thermal equilibrium. We present an alternative approach, closer to the optical case, in which atoms are produced in the same phase space cell by the induced version of a spontaneous process, very far from thermal equilibrium. This process should be the analog of the spontaneous emission of photons,

$$A^* \rightarrow A + \text{photon}. \tag{1}$$

If we replace photons by bosonic atoms B we are led to consider the more general dissociation process,

$$C \rightarrow A + B, \tag{2}$$

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which includes the emission of a photon A as a special case.

In the case of an ordinary laser, the photon energy is provided to the emitter by an external energy source, which excites the atoms. Similarly, in the case of the atomic wave, the energy represented by the mass of the atomic constituents needs also to be provided to the emitters. If we do not want to have to recreate each of these masses individually from other particles, they must also be constituents of the emitter. The simplest solution is therefore to use for the emitter C a composite object (molecule, excited atom or ion) which includes the required boson B as a constituent.

2. Theoretical framework

Like stimulated emission of photons, the enhancement factor in the emission of atoms in the same phase cell by an incoming atomic wave, comes only from the bosonic character and the exchange properties of indistinguishable particles. Following Feynman [1], for n incident atoms B, the statistical weight of the initial state is $1/n!$, the statistical weight of the final state is $1/(n+1)!$, and the amplitude for the process is $(n+1)!$ times the amplitude for spontaneous emission. The probability (per second) of emission is then $[(n+1)!]^2/(n+1)!n! = n+1$ times the probability of spontaneous dissociation. This can be demonstrated rigorously in a quantized field description of the stimulated dissociation process. In such a description, atomic systems in a given internal state are described as elementary particles by quantum field operators [2] satisfying commutation or anti-commutation rules to introduce their bosonic or fermionic character. These quantum fields operate on a Fock space restricted to the positive energy particles sector (we exclude the formation of anti-atoms or anti-molecules from atoms and molecules) and satisfy free-field equations. To take the atomic angular momentum into account, it is necessary to introduce fields for arbitrary spin written as Lorentz spinors or tensors. Here for simplicity, we shall assume scalar (pseudo-scalar) fields which satisfy the Klein–Gordon equation, corresponding to spin zero particles,

$$\phi_\alpha(x) = \frac{1}{(2\pi\hbar)^{3/2}} \int \frac{\mathrm{d}^3 p}{\sqrt{2E_\alpha(\boldsymbol{p})}} c_\alpha(\boldsymbol{p}) \exp\left(\frac{\mathrm{i}}{\hbar}[\boldsymbol{p}\cdot\boldsymbol{r} - E_\alpha(\boldsymbol{p})t]\right), \tag{3}$$

where

$$E_\alpha(\boldsymbol{p}) = (E_\alpha^2 + p^2c^2)^{1/2} \tag{4}$$

is the relativistic energy of the atom in state α and where $E_\alpha/c^2 = M_\alpha$ is the rest mass. The previous expression corresponds to an expansion in plane waves. For many-particle states it is more convenient to introduce sets of discrete modes $v_{\mathrm{B}m}(x)$ such that

$$\phi_{\mathrm{B}}(x) = \sum_m \frac{1}{\sqrt{2V_m E_{\mathrm{B}}(\boldsymbol{p}_{\mathrm{B}m})}} c_{\mathrm{B}m} v_{\mathrm{B}m}(x) \exp\left(\frac{\mathrm{i}}{\hbar}[\boldsymbol{p}_{\mathrm{B}m}\cdot\boldsymbol{r} - E_{\mathrm{B}}(\boldsymbol{p}_{\mathrm{B}m})t]\right). \tag{5}$$

The starting point is then simply an effective Hamiltonian density,

$$\tilde{\mathcal{H}}(x) = g_{\mathrm{R}}\phi_{\mathrm{A}}^\dagger(x)\phi_{\mathrm{B}}^\dagger(x)\phi_{\mathrm{C}}(x) + \mathrm{h.c.}, \tag{6}$$

where the quantum fields $\phi_{\mathrm{A}}^\dagger(x)$, $\phi_{\mathrm{B}}^\dagger(x)$ and $\phi_{\mathrm{C}}(x)$ respectively create atoms A and B and annihilate molecule C at the space–time point x.

Our purpose will be to derive general formulas for the spontaneous dissociation rate and for the gain (A and B Einstein coefficients and line shape) which are applicable both to massive particles and to photons, in order to keep the parallelism between light and matter wave amplifiers and be guided by this analogy. The derivation of such formulas requires a unified approach in which the rest mass of the various partners may be arbitrarily set

Fig. 1. Density matrix diagrams giving respectively (a) the spontaneous decay of C and (b) the time rate of change of the population of species C under the influence of stimulated emission of bosons B.

to zero [1]. A relativistically covariant description of quantum mechanics is obtained if the Schrödinger equation is replaced by the Tomonaga–Schwinger equation. For the density operator $\tilde{\rho}(\sigma)$, the corresponding equation [2] leads to the Liouville–von Neumann equation in a given frame,

$$i\hbar\frac{\partial\tilde{\rho}(t)}{\partial t} = [\tilde{V}(t),\tilde{\rho}(t)] \tag{7}$$

or in integral form

$$\tilde{\rho}(t) = \tilde{\rho}(t_0) + \frac{1}{i\hbar}\int_{t_0}^{t} dt' \,[\tilde{V}(t'),\tilde{\rho}(t')], \tag{8}$$

where

$$\tilde{V}(t) = \int d^3x\, \tilde{\mathcal{H}}(x) \tag{9}$$

is the interaction Hamiltonian.

From these equations one obtains an integral form for the derivative of the density operator,

$$\frac{\partial\tilde{\rho}(t)}{\partial t} = \frac{1}{i\hbar}\int d^3x\,[\tilde{\mathcal{H}}(x),\tilde{\rho}(t_0)] - \frac{1}{\hbar^2}\int d^3x\int d^3x'\int_{t_0}^{t} dt'\,[\tilde{\mathcal{H}}(x),[\tilde{\mathcal{H}}(x'),\tilde{\rho}(t')]]. \tag{10}$$

This equation can be used to derive a master equation for a reduced density matrix $\mathrm{Tr}_B\tilde{\rho}(t)$ with the Markov approximation and to obtain relaxation operators or to derive $\partial\tilde{\rho}(t)/\partial t$ from the zero-order density operator $\tilde{\rho}^{(0)}$ in a second-order perturbation approach to obtain the amplification coefficient.

3. Derivation of the spontaneous dissociation rate

With the Hamiltonian density (6), the trace over the empty modes of B yields the following matrix element of the relaxation operator (see Fig. 1a),

[1] In the case of photons, one should add to field operators (3) their Hermitian conjugate counterpart, but because of the rotating-wave approximation this will not lead to any additional term in the Hamiltonian density.

$$\langle 0_A|\langle 1_{p_C}|\Gamma|1_{p'_C}\rangle|0_A\rangle = \frac{2g_R^2}{\hbar^2}\,\mathrm{Re}\int \mathrm{d}^3x \int \mathrm{d}^3x' \int_{-\infty}^{t} \mathrm{d}t'\, \langle 0_B|\phi_B(x)\phi_B^\dagger(x')|0_B\rangle\langle 0_A|\phi_A(x)\phi_A^\dagger(x')|0_A\rangle$$

$$\times\langle 1_{p_C}|\phi_C^\dagger(x)\phi_C(x')|1_{p'_C}\rangle$$

$$= \Gamma_C\delta(p_C - p'_C), \tag{11}$$

which gives the spontaneous dissociation rate

$$\Gamma_C = \frac{\pi^2 g_R^2}{2h^4 E_C(p_C)}\int\int \frac{\mathrm{d}^3p_A}{E_A(p_A)}\frac{\mathrm{d}^3p_B}{E_B(p_B)}\delta(p_C - p_A - p_B)\delta(E_C(p_C) - E_A(p_A) - E_B(p_B))$$

$$= \frac{1}{\gamma}\frac{2\pi^3 g_R^2}{h^3c^3E_C^2}\frac{p_{B0}c}{h} = \Gamma_{C0}/\gamma, \tag{12}$$

where $\gamma = 1/\sqrt{1 - v_C^2/c^2}$ is the time dilation factor, and where we have introduced the momentum p_{B0} of the emerging particle B in the center-of-mass frame,

$$p_{B0} = \frac{c}{2M_C}[(M_A + M_B + M_C)(-M_A + M_B + M_C)(M_A - M_B + M_C)(-M_A - M_B + M_C)]^{1/2}. \tag{13}$$

From formula (12), we can recover the spontaneous emission rate of photons by setting either M_A or $M_B = 0$ and $p_{B0}c \simeq \hbar\omega_0$ (an additional factor 2/3 would come from spin 1 fields), $\Gamma_{C0} \simeq \mu^2\omega_0^3/2\pi\varepsilon_0\hbar c^3$, where we have used the correspondence between our coupling constant g_R and the transition dipole moment μ, $g_R^2 = 4\mu^2M_C^2c^4(\hbar\omega)^2/\varepsilon_0$. In the case of three non-relativistic massive particles, $p_{B0} \simeq (2\mu_{AB}\Delta E)^{1/2}$ where μ_{AB} is the reduced mass and where $\Delta E/c^2$ is the mass defect.

4. Derivation of the amplification coefficient

The gain for the bosonic wave will be simply obtained from the time rate of change of the average number of bosons in the mode m, by (for simplicity we shall omit the subscript m)

$$\alpha = \frac{1}{v_B}\frac{1}{\bar{n}_B}\frac{\partial\bar{n}_B}{\partial t}, \tag{14}$$

where v_B is the group velocity of that wave. This rate is opposite to the time rate of change of particles C under the influence of stimulated emission of bosons B in a given mode (see Fig. 1b),

$$\alpha = n_C^{(0)}\frac{g_R^2}{\hbar^2}\frac{1}{8E_B(p_B)v_BE_C^2} \times 2\,\mathrm{Re}\int_0^\infty \mathrm{d}\tau \int \mathrm{d}^3p_C \frac{E_C^2}{E_C(p_C)E_A(p_C - p_B)}$$

$$\times \exp\{\mathrm{i}[E_C(p_C) - E_A(p_C - p_B) - E_B(p_B)]\tau/\hbar - \Gamma\tau/2\}F(p_C), \tag{15}$$

where $n_C^{(0)}F(p_C)$ is the zero-order population of species C, with a normalized momentum distribution $F(p_C)$ of width M_Cu.

The coupling constant can be eliminated in favor of the decay constant,

$$\frac{g_R^2}{\hbar^2}\frac{1}{8E_B(p_B)v_BE_C^2} = \frac{h^2}{p_Bp_{B0}}\frac{\Gamma_{C0}}{4\pi}.$$

Like in the case of light amplification, the gain may thus be either homogeneously or inhomogeneously broadened, depending on the relative size of $\hbar\Gamma$ and of the inhomogeneous energy width. This inhomogeneous

width is of the order of the spread in kinetic energies ($\sim M_C u^2$) in the case of massive particles but it is much larger ($\sim M_C uc$) when $M_A = 0$.

We will give expressions for the gain in the limits of a purely homogeneously or inhomogeneously broadened line shape. In the inhomogeneous case the τ integral gives a δ function for the energy and if the distribution $F(\boldsymbol{p}_C)$ is a non-relativistic Maxwell–Boltzmann distribution of width $M_C u$ we obtain

$$\alpha(p_B) = \frac{n_C^{(0)}}{4\pi^{3/2}} \frac{h^3}{p_B^2 p_{B0}} \frac{\Gamma_{C0}}{u} \sqrt{\pi} \frac{c}{u} \exp(c^2/u^2) (\mathrm{erf}\{(c/u)[E_B(\boldsymbol{p}_{B0}) E_B(\boldsymbol{p}_B) + p_{B0} p_B c^2]/E_B^2\}$$

$$-\mathrm{erf}\{(c/u)[E_B(\boldsymbol{p}_{B0}) E_B(\boldsymbol{p}_B) - p_{B0} p_B c^2]/E_B^2\}), \tag{16}$$

which gives again the usual Doppler profile when M_B vanishes. When the boson mass is different from zero, the previous formula simplifies in the non-relativistic approximation,

$$\alpha(p_B) = \frac{n_C^{(0)}}{4\pi^{3/2}} \frac{h^3}{p_B^2 p_{B0}} \frac{\Gamma_{C0}}{u} \left[\exp\left(- \frac{(p_B - p_{B0})^2}{M_B^2 u^2} \right) - \exp\left(- \frac{(p_B + p_{B0})^2}{M_B^2 u^2} \right) \right]. \tag{17}$$

Both (16) and (17) exhibit a resonance for $p_B \simeq p_{B0}$, if p_{B0} is large enough compared to $M_B u$. We can also extract the Einstein B coefficient,

$$B_\nu = h^2 c \Gamma_{C0} / 4\pi p_B^2 E_B \tag{18}$$

associated with a line shape normalized in frequency $\nu = p_B c/h$. If $M_B u$ is large compared to p_{B0} then

$$\alpha(p_B) \simeq \frac{n_C^{(0)}}{\pi^{3/2}} \left(\frac{h}{M_B u} \right)^3 \frac{\Gamma_{C0}}{v_B}. \tag{19}$$

Let us point out that, from (12) and (15), one can show that, in the non-relativistic limit, the following relation should be satisfied between the gain and the dissociation constant,

$$4\pi \int_0^\infty v_B p_B^2 \alpha(p_B)\, \mathrm{d}p_B = n_C^{(0)} h^3 \Gamma_{C0}$$

and one can check that this is indeed satisfied by (17). This relation is the analog of the Füchtbauer–Ladenburg formula in optical spectroscopy.

The homogeneous gain formula is obtained in the limit where $F(\boldsymbol{p}_C)$ is considered as a δ function in (15),

$$\alpha(p_B) = n_C^{(0)} \frac{h^2}{p_B p_{B0}} \frac{E_C}{E_A(-\boldsymbol{p}_B)} \frac{1}{\pi} \frac{\Gamma\Gamma_{C0}/4}{(\Gamma/2)^2 + [E_C - E_A(-\boldsymbol{p}_B) - E_B(\boldsymbol{p}_B)]^2/\hbar^2}. \tag{20}$$

For $M_B = 0$ we recover the usual formula for light amplification with the natural width for $\Gamma = \Gamma_{C0} = A$,

$$\alpha(\omega) = n_C^{(0)} \lambda_0^2 \frac{1}{2\pi} \frac{(A/2)^2}{(A/2)^2 + (\omega - \omega_0)^2} \tag{21}$$

(the factor 2 in the denominator comes from a proper account of the polarization of light). For $M_A = 0$, the homogeneous limit is obtained only in the extreme case $\hbar\Gamma > M_C uc$,

$$\alpha(p_B) = n_C^{(0)} \lambda_0^2 \frac{c}{v_B} \frac{1}{\pi} \frac{(A/2)^2}{(A/2)^2 + [E_C - E_B(\boldsymbol{p}_B) - p_B c]^2/\hbar^2}. \tag{22}$$

For $M_A, M_B, M_C \neq 0, \Gamma = \Gamma_{C0}$,

$$\alpha(p_B) = n_C^{(0)} \frac{h^2}{p_B p_{B0}} \frac{M_B}{\mu_{AB}} \frac{1}{\pi} \frac{(\Gamma/2)^2}{(\Gamma/2)^2 + [(p_{B0}^2 - p_B^2)/2\mu_{AB}]^2/\hbar^2}. \tag{23}$$

We see that, at resonance, the cross section is again proportional to a de Broglie wavelength squared, and therefore one should use atoms as slow as possible. The mass defect $\Delta E/c^2$ should thus be as small as possible.

5. Conclusion and possible gain media

We get formulas for atomic waves which are similar to the optical case. This demonstrates that comparable gains could be obtained with the same wavelengths and population inversion densities. It should be noted, however, that a divergent gain occurs for vanishing atomic velocity and this could be in favor of the slowest atomic waves for a given length of amplifier and fixed resonator losses.

In Eq. (1) species B is an atomic or a molecular boson, but A can be anything from a photon to a solid, i.e., C can be an excited atom, a negative ion, a molecule, a cluster or even a surface (B may in fact be trapped in any external potential). Especially interesting possibilities are the predissociation of electronically excited molecules (e.g. $Na_2(B^1\Pi_u) \rightarrow Na(3p) + Na(3s)$) or of vibrationally excited van der Waals molecules (e.g. $He–I_2$) or the dissociative recombination of ions (e.g. $HeH^+ + e^- \rightarrow HeH \rightarrow He + H$).

The only requirement to maintain the population inversion is that species A should escape or be removed as fast as possible through spontaneous decay, optical pumping or any other means. Finally losses by elastic and inelastic collisions of B with A or C need to be investigated in each case.

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