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CONDENSATS DE BOSE–EINSTEIN ET LASERS À ATOMES
BOSE–EINSTEIN CONDENSATES AND ATOM LASERS

Theoretical tools for atom optics and interferometry

Christian J. BORDÉ [a,b]

[a] **Laboratoire de physique des lasers, UMR 7538 CNRS, université Paris-Nord, 99 avenue J.-B. Clément, Villetaneuse, 93430 France**

[b] **Institut für Quantenoptik, Universität Hannover, Welfengarten 1, 30167 Hannover, Germany**
E-mail: borde@lpl.univ-paris13.fr

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Abstract. The development of high sensitivity and high accuracy atom interferometers requires new theoretical tools for their modelization: in this article we emphasize specifically a generalized Fresnel–Kirchhoff formula for atom optics in the form of $ABCD$ matrices and covariant wave equations in the form of a Dirac equation for atom optics in the presence of gravito-inertial fields. As examples, we derive the phase shift for the atom gravimeter and the output of an atom laser. Some of the physics of the beam splitters is described. We present a second-quantized field theory of massive spin one-half particles or antiparticles in the presence of a weak gravitational field treated as a spin two external field in a flat Minkowski background. This theory is used to calculate and discuss relativistic phase shifts in the context of matter-wave interferometry (especially atom or antiatom interferometry). In this way, many effects are introduced in a unified relativistic framework, including spin-gravitation terms: gravitational red shift, Thomas precession, Sagnac effect, spin-rotation effect, orbital and spin Lense–Thirring effects, de Sitter geodetic precession and finally the effect of gravitational waves. © 2001 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

atom interferometer / Gaussian atom optics / ABCD matrices / covariant wave equations / atom gravimeter / gravito-inertial fields / atom laser / relativistic phase shifts / antihydrogen / Ramsey fringes

Outils théoriques pour l'optique et l'interférométrie atomiques

Résumé. *Le développement d'interféromètres atomiques de haute sensibilité et de grande exactitude requiert aujourd'hui de nouveaux outils théoriques permettant leur modélisation précise. Dans ce cours, nous mettons l'accent spécifiquement, d'abord sur une formule de Fresnel–Kirchhoff généralisée déclinée sous la forme de matrices $ABCD$ et ensuite, sur des équations d'onde covariantes pour l'optique atomique en présence de champs gravito-inertiels, écrites sous la forme d'une équation de Dirac pour chaque niveau d'énergie interne des atomes. Les matrices $ABCD$ sont introduites à trois dimensions, d'abord dans un cadre classique par l'équation de Hamilton–Jacobi puis, dans un cadre quantique, au moyen du propagateur de Van Vleck. Le déphasage d'un gravimètre atomique et le mode de sortie d'un laser à atomes sont calculés à titre d'exemples. La physique des séparatrices atomiques n'est que partiellement décrite. Elle l'est essentiellement pour souligner la réinterprétation de l'origine des franges de Ramsey qu'impose la description quantique du mouvement externe des atomes et pour illustrer la difficulté du chemin vers le calcul rigoureux complet du signal de franges d'un interféromètre. On termine ce cours par la théorie des particules ou des antiparticules massives de spin $1/2$, en deuxième*

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quantification, en présence d'un champ gravitationnel faible considéré comme champ extérieur de spin 2 dans un espace–temps plat de Minkowski. Cette théorie est utilisée pour calculer et discuter les déphasages relativistes dans le contexte général de l'interférométrie à ondes de matière et plus spécialement à ondes atomiques ou antiatomiques. De cette façon, beaucoup d'effets sont introduits dans un cadre relativiste unifié, incluant les termes d'interaction entre spin et gravitation : décalage gravitationnel vers le rouge, précession de Thomas, effet Sagnac, effet de spin-rotation, effets Lense–Thirring orbital et de spin, précession géodésique de de Sitter et finalement effet des ondes gravitationnelles.

interféromètre atomique / optique atomique gaussienne / matrices ABCD / équations d'ondes covariantes / gravimètre atomique / champs gravito-inertiels / laser à atomes / déphasages relativistes / antihydrogène / frange de Ramsey

1. Introduction

Atom interferometers have become remarkable measuring instruments when used as clocks in the microwave or in the optical domain, as inertial sensors or for the determination of atomic masses and of the fine structure constant [1–11]. The accuracy of these devices is now such that a new theoretical framework is required, which includes:

(1) a fully quantum mechanical treatment of the atomic motion in free space and in the presence of a gravitational field (most cold atom interferometric devices use atoms in 'free fall' in a fountain geometry, e.g. the Cesium atomic clock or the Cesium gravimeter);
(2) an account of simultaneous actions of gravitational and electromagnetic fields in beam splitters;
(3) a second quantization of the matter fields to take into account their fermionic or bosonic character in order to discuss the role of coherent sources and their noise properties;
(4) a covariant treatment including spin to evaluate general relativistic effects [10,12,13].

One would also like to be able to discuss more futuristic applications like the propagation of antimatter in interferometers in the presence of gravitation and the properties of coherent antimatter waves (generated by an antiatom laser such as an antihydrogen Bose–Einstein condensate) or like the physics of active matter(antimatter)-wave gyros using matter(antimatter)-wave amplifiers [14–16].

The current state-of-the-art of our theoretical description of atom interferometry to meet these requirements is presented. To derive accurate formulas for the phase shifts, the propagation of atom fields using the $ABCD$ formalism of Gaussian optics is reviewed [17,18]. Specific applications to atom gravimeters and to the output of atom lasers are considered for the illustration. Concerning the application to atomic microwave or optical clocks, the quantization of the atomic motion in the longitudinal direction leads to a reinterpretation of Ramsey fringes, in terms of a velocity change in this direction, as well as in the transverse direction [4,9,10]. The dispersion surface of the beam splitters is then discussed, from which a new look at Rabi oscillations as a pendellösung phenomenon is developed [10,19]. Finally, relativistic phase shifts are recalled, using quantum field theory of atoms with spin, in the presence of weak gravitational fields treated as spin-2 fields in flat space–time [13].

2. Non-relativistic approach

We shall consider quite generally the non-relativistic Schrödinger equation as the non-relativistic limit of a general relativistic equation described in the second part of this paper:

$$\mathrm{i}\hbar\frac{\partial|\Psi(t)\rangle}{\partial t} = \Big[H_0 + \frac{1}{2M}\vec{p}_{\mathrm{op}}\cdot\vec{\vec{g}}(t)\cdot\vec{p}_{\mathrm{op}} - \vec{\Omega}(t)\cdot\big(\vec{L}_{\mathrm{op}}+\vec{S}_{\mathrm{op}}\big)$$

$$- M\vec{g}(t)\cdot\vec{r}_{\rm op} - \frac{M}{2}\vec{r}_{\rm op}\cdot\overset{\Rightarrow}{\gamma}(t)\cdot\vec{r}_{\rm op} + V(\vec{r}_{\rm op},t)\Big]\big|\Psi(t)\big\rangle \tag{1}$$

where H_0 is an internal atomic Hamiltonian and $V(\vec{r}_{\rm op},t)$ some general interaction Hamiltonian with an external field. Gravito-inertial fields are represented by the tensors $\overset{\Rightarrow}{g}(t)$ and $\overset{\Rightarrow}{\gamma}(t)$ and by the vectors $\vec{\Omega}(t)$ and $\vec{g}(t)$. The same terms can also be used to represent the effect of various external electromagnetic fields. The operators $\vec{L}_{\rm op} = \vec{r}_{\rm op}\times\vec{p}_{\rm op}$ and $\vec{S}_{\rm op}$ are respectively the orbital and spin angular momentum operators. Apart from in $V(\vec{r}_{\rm op},t)$ we have limited the dependence of the Hamiltonian to second-order in the operators $\vec{p}_{\rm op}$ and $\vec{r}_{\rm op}$.

We shall use a general approach based upon a time-dependent description of the evolution of wave packets. The basis of this treatment is the use of the interaction representation with respect to $H_0 - \vec{\Omega}(t)\cdot\vec{S}_{\rm op}$ and to those parts of the Hamiltonian which are connected only with the external motion:

$$\big|\widetilde{\Psi}(t)\big\rangle = U_0^{-1}(t,t_1)\big|\Psi(t)\big\rangle, \quad \text{where} \tag{2}$$

$$U_0(t,t_1) = U_{\rm E}(t,t_1)\exp\big[-{\rm i}H_0(t-t_1)/\hbar\big]\mathcal{T}\exp\left[\frac{\rm i}{\hbar}\int_{t_1}^{t}\vec{\Omega}(t')\cdot\vec{S}_{\rm op}\,{\rm d}t'\right] \tag{3}$$

is the free evolution operator in the absence of V, in which $U_{\rm E}(t,t_1)$ describes the center-of-mass motion and where $\mathcal{T}$ is a time-ordering operator.

The transformed ket$|\widetilde{\Psi}(t)\rangle$ satisfies:

$$ {\rm i}\hbar\partial_t\big|\widetilde{\Psi}(t)\big\rangle = \widetilde{V}(\vec{r}_{\rm op},\vec{p}_{\rm op},t)\big|\widetilde{\Psi}(t)\big\rangle \quad \text{with} \tag{4}$$

$$\widetilde{V}(\vec{r}_{\rm op},\vec{p}_{\rm op},t) = U_0^{-1}(t,t_1)V(\vec{r}_{\rm op},t)U_0(t,t_1) = \widehat{V}\big(\vec{R}_{\rm op}(t,t_1),t\big) \quad \text{and} \tag{5}$$

$$\vec{R}_{\rm op}(t,t_1) = U_{\rm E}^{-1}(t,t_1)\vec{r}_{\rm op}U_{\rm E}(t,t_1) \tag{6}$$

For expressions for the evolution operators $U_0(t,t_1)$ in the presence of inertial fields see [3,17] and below. In free space $U_{\rm E}(t,t_1)$ reduces to $\exp[-{\rm i}(\vec{p}_{\rm op}^{\,2}/2M)(t-t_1)/\hbar]$, in which case:

$$\vec{R}_{\rm op}(t,t_1) = \vec{r}_{\rm op} + \vec{p}_{\rm op}(t-t_1)/M \tag{7}$$

The formal solution of equation (4) is:

$$\big|\widetilde{\Psi}(t)\big\rangle = \widetilde{U}(t,t_0)\big|\widetilde{\Psi}(t_0)\big\rangle = \mathcal{T}\exp\left[-\frac{\rm i}{\hbar}\int_{t_0}^{t}{\rm d}t'\,\widehat{V}\big(\vec{R}_{\rm op}(t',t_1),t'\big)\right]\big|\widetilde{\Psi}(t_0)\big\rangle \tag{8}$$

This formula gives the evolution in the beam splitters or describes the effect of any additional external perturbation on the interferometer arms. One way to handle the calculation of this expression is to replace $\vec{R}_{\rm op}(t',t_1)$ with the classical trajectory. This is the basis for the calculation of the beam splitters given in reference [20]. We shall see below some other approaches based on exact calculations.

The general idea is then to write the evolution operator $U(t,t_0)$ as a product of free evolution operators and of $\widetilde{S}$-matrices which give the evolution in the beam splitters. From:

$$U(t,t_0) = U_0(t,t_1)U_0^{-1}(t,t_1)U(t,t_0)U_0^{-1}(t_1,t_0)U_0(t_1,t_0) \tag{9}$$

$$= U_0(t,t_1)\widetilde{U}(t,t_0)U_0(t_1,t_0) \tag{10}$$

we obtain:

$$\big\langle\vec{r}\big|\Psi(t)\big\rangle = \int {\rm d}^3r_1\,{\rm d}^3r_0\,\mathcal{K}_{\rm T}(\vec{r},\vec{r}_1,t,t_1)\widetilde{S}(\vec{r}_1,t_1)\mathcal{K}_{\rm T}(\vec{r}_1,\vec{r}_0,t_1,t_0)\big\langle\vec{r}_0\big|\Psi(t_0)\big\rangle \tag{11}$$

where $\mathcal{K}_{\mathrm{T}}(\vec{r},\vec{r}\,',t,t') = \langle \vec{r}|U_0(t,t')|\vec{r}\,'\rangle$ is the propagator outside the field zones or in the absence of the perturbation V and where the $\widetilde{S}$-matrix is given by:

$$\widetilde{S}(\vec{r},t) = \mathcal{T}\exp\left[-\frac{\mathrm{i}}{\hbar}\int_{-\infty}^{+\infty}\mathrm{d}t'\,\widehat{V}\big(\vec{R}(t',t),t'\big)\right] \tag{12}$$

where $\vec{R}(t',t)$ is the classical atomic trajectory leading to $\vec{R}(t,t)=\vec{r}$ and with the proper recoil correction for each wave packet as explained in [20]. Expressions for this matrix can be found in [9,20] and expressions for the propagators in the presence of inertial fields can be found in [3,17] and will be rederived below.

Equivalently, we can write an integral equation for $|\,\widetilde{\Psi}(t)\rangle$:

$$\big|\,\widetilde{\Psi}(t)\big\rangle = \big|\,\widetilde{\Psi}(t_0)\big\rangle + \frac{1}{\mathrm{i}\hbar}\int_{t_0}^{t}\mathrm{d}t'\,\widetilde{V}(\vec{r}_{\mathrm{op}},\vec{p}_{\mathrm{op}},t')\big|\,\widetilde{\Psi}(t')\big\rangle \tag{13}$$

which can be iterated perturbatively. The corresponding integral equation for $|\,\Psi(t)\rangle$ is:

$$\big|\,\Psi(t)\big\rangle = U(t,t_0)\big|\,\Psi(t_0)\big\rangle + \frac{1}{\mathrm{i}\hbar}\int_{t_0}^{t}\mathrm{d}t'\,U(t,t')V(\vec{r}_{\mathrm{op}},t')\big|\,\Psi(t')\big\rangle \tag{14}$$

and in the $\langle\vec{r}|$ representation, letting $t_0\to-\infty$:

$$\Psi(\vec{r},t) = \Psi^{(0)}(\vec{r},t) + \frac{1}{\mathrm{i}\hbar}\int_{-\infty}^{+\infty}\mathrm{d}t'\int\mathrm{d}^3r'\,\mathcal{G}_{\mathrm{T}}(\vec{r},\vec{r}\,',t,t')V(\vec{r}\,',t')\,\Psi(\vec{r}\,',t') \tag{15}$$

where $\Psi^{(0)}(\vec{r},t)=\int\mathrm{d}^3r_0\,\mathcal{K}_{\mathrm{T}}(\vec{r},\vec{r}_0,t,-\infty)\,\Psi(\vec{r}_0,-\infty)$ is the wavefunction in the absence of the perturbation V assumed to vanish at $-\infty$ and where:

$$\mathcal{G}_{\mathrm{T}} = \theta(t-t')\mathcal{K}_{\mathrm{T}} \tag{16}$$

is the Green function, related to the propagator $\mathcal{K}_{\mathrm{T}}$ through the Heaviside step function θ.

Stationary case

If the interaction Hamiltonian $V(\vec{r},t)$ can be transformed into a time-independent operator $\widehat{V}(\vec{r})$, e.g. by using the rotating-wave approximation in the rotating (internal) frame [9,10], then we can obtain the wavefunction:

$$\widehat{\Psi}_{\mathrm{E}}(\vec{r}) = \widehat{\Psi}_{\mathrm{E}}^{(0)}(\vec{r}) + \frac{1}{\mathrm{i}\hbar}\int\mathrm{d}^3r'\,\widehat{\mathcal{G}}_{\mathrm{E}}(\vec{r},\vec{r}\,')\widehat{V}(\vec{r}\,')\,\widehat{\Psi}_{\mathrm{E}}(\vec{r}\,') \tag{17}$$

from the Fourier transform of (15), thanks to a time-independent Green function:

$$\widehat{\mathcal{G}}_{\mathrm{E}}(\vec{r},\vec{r}\,') = \int_0^{+\infty}\mathrm{d}t_1\,\mathrm{e}^{\mathrm{i}Et_1/\hbar}\mathcal{K}_{\mathrm{T}}(\vec{r},\vec{r}\,',t_1) \tag{18}$$

We shall give below a specific example of such a time-independent Green function in the case of a gravitational field.

2.1. Explicit form of the propagator: ABCD matrices

This propagator is derived in great detail in reference [17]. The idea is to use a series of unitary transformations which eliminate successively the various terms of equation (1). As an example, to eliminate the rotation term $-\overrightarrow{\Omega}(t)\cdot(\overrightarrow{L}_{\mathrm{op}}+\overrightarrow{S}_{\mathrm{op}})$, we perform the unitary transformation:

$$\big|\Psi(t)\big\rangle = U_\Omega \big|\Psi_0(t)\big\rangle \quad \text{with} \tag{19}$$

$$U_\Omega(t,t_0) = \mathcal{T}\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\vec{\Omega}(t_1)\cdot\vec{L}_{\mathrm{op}}\,\mathrm{d}t_1\right]\mathcal{T}\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\vec{\Omega}(t_1)\cdot\vec{S}_{\mathrm{op}}\,\mathrm{d}t_1\right] \tag{20}$$

We obtain the following equation for the state vector in the 'non-rotating' frame:

$$\mathrm{i}\hbar\frac{\partial|\Psi_0(t)\rangle}{\partial t} = \left[H_0 + \frac{1}{2M}\vec{p}_{\mathrm{op}}\cdot\vec{\vec{h}}_0(t)\cdot\vec{p}_{\mathrm{op}} - M\vec{g}_0(t)\cdot\vec{r}_{\mathrm{op}} - \frac{M}{2}\vec{r}_{\mathrm{op}}\cdot\vec{\vec{\gamma}}_0(t)\cdot\vec{r}_{\mathrm{op}} + V_0(\vec{r}_{\mathrm{op}},t)\right]\big|\Psi_0(t)\big\rangle \tag{21}$$

For simplicity, in the following, we shall omit the subscript 0 for the 'non-rotating' frame. As shown in detail in reference [17], a new series of unitary transformations:

$$\big|\widetilde{\Psi}(t)\big\rangle = U^{-1}\big|\Psi(t)\big\rangle \quad \text{where} \tag{22}$$

$$U(t,t_0) = U_1(t,t_0)\dots U_6(t,t_0)\exp\big[-\mathrm{i}H_0(t-t_0)/\hbar\big] \tag{23}$$

eliminates one term after the other and brings equation (21) to the required form (4):

$$\mathrm{i}\hbar\frac{\partial|\widetilde{\Psi}(t)\rangle}{\partial t} = \widetilde{V}(\vec{r}_{\mathrm{op}},\vec{p}_{\mathrm{op}},t)\big|\widetilde{\Psi}(t)\big\rangle \tag{24}$$

Here, in order to obtain the propagator, we shall use a shortcut through the classical limit and use a well-known result of Van Vleck to make the connection with Quantum Mechanics.

2.1.1. *ABCD matrices in the classical limit*

Let us consider first the external motion of a point particle in a time-dependent gravito-inertial or electromagnetic field characterized by $\vec{\vec{g}}(t)$, $\vec{g}(t)$, $\vec{\vec{\gamma}}(t)$, $\vec{\Omega}(t), V(t)$ and for which the classical Hamiltonian is:

$$H = \frac{1}{2M}\vec{p}\cdot\vec{\vec{g}}(t)\cdot\vec{p} - \vec{\Omega}(t)\cdot\vec{L} - M\vec{g}(t)\cdot\vec{r} - \frac{M}{2}\vec{r}\cdot\vec{\vec{\gamma}}(t)\cdot\vec{r} + V(t) \tag{25}$$

Then, the Hamilton's principal function is also at most quadratic in $q=(x,y,z)$ and $q'=(x',y',z')$:

$$S(q,t,q',t') = a + \tilde{b}q + \tilde{c}q' + \frac{M}{2}\big[\tilde{q}DB^{-1}q - 2\tilde{q}\widetilde{B^{-1}}q' + \tilde{q}'B^{-1}Aq'\big] \tag{26}$$

where the matrices $A,\ B,\ D$ have been introduced so that:

$$p = \nabla_q S = b + MDB^{-1}q - M\widetilde{B^{-1}}q', \qquad -p' = \nabla_{q'}S = c - MB^{-1}q + MB^{-1}Aq' \tag{27}$$

is consistent with the $ABCD$ tranformation law:

$$\begin{pmatrix} q \\ p/M \end{pmatrix} = \begin{pmatrix} Bc/M \\ (b+Dc)/M \end{pmatrix} + \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} q' \\ p'/M \end{pmatrix} \quad \text{with} \tag{28}$$

$$\widetilde{B^{-1}} = DB^{-1}A - C \tag{29}$$

From the Hamilton–Jacobi equation:

$$\frac{\partial S}{\partial t} + H(q,\nabla_q S,t) = 0 \tag{30}$$

the following equations are obtained for $a,\ b,\ c,\ A,\ B,\ C,\ D$:

$$\begin{aligned}
&\dot{a}+\frac{1}{2M}\tilde{b}n^{-1}b+V=0\\
&\dot{b}+DB^{-1}n^{-1}b-Mg+\mathrm{i}(J\cdot\Omega)b=0\\
&\dot{c}-B^{-1}n^{-1}b=0
\end{aligned}\tag{31}$$

$$\begin{aligned}
&\dot{A}+\mathrm{i}(J\cdot\Omega)A=n^{-1}C\\
&\dot{B}+\mathrm{i}(J\cdot\Omega)B=n^{-1}D\\
&\dot{C}+\mathrm{i}(J\cdot\Omega)C=\gamma A\\
&\dot{D}+\mathrm{i}(J\cdot\Omega)D=\gamma B
\end{aligned}\tag{32}$$

where $n^{-1},\ \gamma,\ g,\ \Omega$ are matrices corresponding to the tensors $\vec{\vec{g}}(t),\ \vec{\vec{\gamma}}(t)$ and vectors $\vec{g}(t),\ \vec{\Omega}(t)$ and where J are the rotation matrices.

One can then perform a coordinate transformation $q\to q_0$, which corresponds to a common rotation leaving S invariant, to get rid of all the terms involving $J\cdot\Omega$:

$$\begin{aligned}
(q,g)&=\mathcal{U}(t,t')(q_0,g_0)\\
\left(n^{-1},\gamma\right)&=\mathcal{U}(t,t')\left(n_0^{-1},\gamma_0\right)\mathcal{U}^{-1}(t,t')\\
(A,B,C,D,b)&=\mathcal{U}(t,t')(A_0,B_0,C_0,D_0,b_0)
\end{aligned}\tag{33}$$

with the following orthogonal rotation matrix:

$$\mathcal{U}(t,t')=\mathcal{T}\exp\left(-\mathrm{i}\int_{t'}^{t}J\cdot\Omega(t_1)\,\mathrm{d}t_1\right)\tag{34}$$

The solution of the first three equations is then (dropping the subscript 0 everywhere):

$$\begin{aligned}
a&=-M\tilde{\dot{\xi}}n\xi+\frac{M}{2}\tilde{\xi}DB^{-1}\xi+\frac{M}{2}\int_{t'}^{t}\left(\tilde{\dot{\xi}}n\dot{\xi}+\tilde{\xi}\gamma\xi+2\tilde{g}\xi\right)\mathrm{d}t_1-\int V\,\mathrm{d}t_1\\
b&=M\left(n\dot{\xi}-DB^{-1}\xi\right)\\
c&=MB^{-1}\xi
\end{aligned}\tag{35}$$

where ξ satisfies:

$$\ddot{\xi}+n^{-1}\dot{n}\dot{\xi}-n^{-1}\gamma\xi-n^{-1}g=0\tag{36}$$

If n^{-1}, γ and g are independent of time:

$$\xi=\gamma^{-1}\left[1-\cosh\left[\left(\gamma n^{-1}\right)^{1/2}\left(t-t'\right)\right]\right]g\tag{37}$$

From the system of equations (32), the $ABCD$ matrix satisfies:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}A&B\\C&D\end{pmatrix}=\begin{pmatrix}0&n^{-1}\\\gamma&0\end{pmatrix}\begin{pmatrix}A&B\\C&D\end{pmatrix}\tag{38}$$

When n^{-1} and γ commute, this equation is trivially integrated with the $SL(2C)$ group parametrization formula, by introducing dimensionless quantities with an arbitrary t_1 and the Pauli matrices $\vec{\sigma}$, as in [17]. If n^{-1} and γ are independent of time:

$$\begin{pmatrix}A&B/t_1\\Ct_1&D\end{pmatrix}=\sigma_0\cosh\left[\left(n^{-1}\gamma\right)^{1/2}(t-t')\right]+\vec{\sigma}\cdot\hat{u}\sinh\left[\left(n^{-1}\gamma\right)^{1/2}(t-t')\right]\tag{39}$$

where $\hat{u} = ((n^{-1/2}\gamma^{-1/2}/t_1 + \gamma^{1/2}n^{1/2}t_1)/2, \mathrm{i}(n^{-1/2}\gamma^{-1/2}/t_1 - \gamma^{1/2}n^{1/2}t_1)/2, 0)$ satisfies $\hat{u}^2 = I$, and where I and σ_0 are respectively (3×3) and (2×2) unit matrices.

The $ABCD$ law reads:

$$\begin{pmatrix} q \\ p/M \end{pmatrix} = \begin{pmatrix} \xi \\ n\dot{\xi} \end{pmatrix} + \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} q' \\ p'/M \end{pmatrix} \tag{40}$$

From these expressions we are now able to write the classical action integral:

$$\begin{aligned} S(q,t,q',t') = {} & M\dot{\tilde{\xi}}n(q-\xi) + \int_{t'}^{t} L(t_1)\,\mathrm{d}t_1 - \int V\,\mathrm{d}t \\ & + \frac{M}{2}\big[(\tilde{q}-\tilde{\xi})DB^{-1}(q-\xi) - 2(\tilde{q}-\tilde{\xi})\widetilde{B^{-1}}q' + \tilde{q}'B^{-1}Aq'\big] \end{aligned} \tag{41}$$

where $L = M(\dot{\tilde{\xi}}n\dot{\xi} + \tilde{\xi}\gamma\xi + 2\tilde{g}\xi)/2$ is a partial Lagrangian. Finally the rotation (33) is used to bring back q in the rotating frame.

2.1.2. *Quantum-mechanical form of the ABCD law*

The quantum-mechanical propagator is then readily obtained from Van Vleck's formula [21]:

$$\mathcal{K}(q,t,q',t') = \pm\left(\frac{1}{2\pi\,\mathrm{i}\hbar}\right)^{N/2}\left|\det\frac{\partial^2 S}{\partial q_i \partial q_j'}\right|^{1/2}\exp[\mathrm{i}S/\hbar] \tag{42}$$

where N is the dimension of space. For three space dimensions $q = (x,y,z)$:

$$\mathcal{K}(q,t,q',t') = \left(\frac{M}{2\pi\,\mathrm{i}\hbar}\right)^{3/2}|\det B|^{-1/2}\exp[\mathrm{i}S/\hbar] \tag{43}$$

where S is the action integral given above.

Then, we use the following formula (and its derivatives) to calculate the action of this propagator on a wave packet [22]:

$$\frac{1}{(2\pi)^{N/2}}\int \exp\left(\frac{\mathrm{i}}{2}\tilde{x}Qx\right)\exp\left(-\mathrm{i}\tilde{\xi}x\right)\mathrm{d}x = \exp(\mathrm{i}\varphi)|\det Q|^{-1/2}\exp\left(-\frac{\mathrm{i}}{2}\tilde{\xi}Q^{-1}\xi\right) \tag{44}$$

where φ is a constant fixed by the N value. When this propagator is applied to a complete basis of Hermite–Gauss wave packets in each dimension, the following result is obtained:

$$\mathit{wave\_packet}(q,t) = \exp\left[\frac{\mathrm{i}S_{\mathrm{cl}}(t,t_0)}{\hbar}\right]\mathit{wave\_packet}_{@t_0}(q-q_{\mathrm{cl}}, v_{\mathrm{cl}}, X, Y) \tag{45}$$

where $S_{\mathrm{cl}}(t,t_0)$ is the classical action and where $\mathit{wave\_packet}_{@t_0}$ is the wave packet at the initial time t_0 e.g. the lowest-order Gaussian mode:

$$\frac{1}{\sqrt{\det X_0}}\exp\left[\frac{\mathrm{i}M}{2\hbar}(\tilde{q}-\widetilde{q_0})Y_0X_0^{-1}(q-q_0)\right]\exp\big[\mathrm{i}M\widetilde{v_0}(q-q_0)/\hbar\big] \tag{46}$$

in which the central position q_0, the initial velocity v_0 and the initial complex width parameters X_0, Y_0 in phase space (all of which are matrices) have to be replaced by their values at time t given by the A,B,C,D transformation law:

$$q_{\mathrm{cl}} = Aq_0 + Bv_0 + \xi, \qquad X = AX_0 + BY_0$$
$$v_{\mathrm{cl}} = Cq_0 + Dv_0 + \dot{\xi}, \qquad Y = CX_0 + DY_0 \tag{47}$$

Each wave packet develops therefore an overall phase which is given by the action integral between two instants. But, in a given interferometer, the two wave packets originating from the same initial wave packet will generally not be recombined with their centers at the same space–time point, with exactly the same velocity or even the same curvature and size. Thus, between these two wave packets, we shall have also a spatial phase corresponding to the distance between their respective centers or to their relative velocities in the factors $\exp[\mathrm{i}M\widetilde{v_{\mathrm{cl}}}(q - q_{\mathrm{cl}})/\hbar]$ and finally to their complex curvature in the Gaussian envelope. Generally, it will be necessary to calculate the detection signal as an integral over space taking into account the partial overlap and the coherence of the wave packets.

The main interest of this formalism is to provide a framework in which all phase shifts due to the terms of the Hamiltonian with a degree $\leqslant 2$ are built in, and thus, calculated exactly to all orders. The contribution of any additional term will be calculated to first order through a time integral over the classical trajectory defined by the $ABCD\xi$ framework.

Conversely, we may extract from this $ABCD\xi$ framework a small piece of the Hamiltonian and view it as a perturbation along the unperturbed trajectory calculated with the $ABCD\xi$ formalism without this term. As an example, let us consider the phase created by the recoil due to the interaction with laser beams. The combination of the two types of terms (action and spatial phase) gives a total phase shift equal to the time integral of the corresponding kinetic energy along the undeflected trajectories. This shows the direct link between the recoil shift and the extra-kinetic energy accumulated along one of the arms with respect to the other arm of the interferometer, and helps to understand why this quantity must be maximized to have a large recoil shift in order to measure the quantum of circulation $\hbar/M$ [23].

2.2. Examples of applications

In order to illustrate the previous formalism, we shall limit ourselves for pedagogical reasons to one dimensional problems:

$$\vec{g}_0(z) = -(g - \gamma z/2)\hat{z}$$

The propagator simplifies to [17]:

$$\mathcal{K}_{\mathrm{T}}(z, z', t, t') = \exp\bigl[-\mathrm{i}H_0(t - t')/\hbar\bigr] \exp\left[\frac{\mathrm{i}M}{\hbar}\dot{\xi}(z - \xi)\right] \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t} L(t_1)\,\mathrm{d}t_1\right]\mathcal{K}(z - \xi, z', t, t') \tag{48}$$

where $\mathcal{K}$ is the propagator $(M/2\pi\,\mathrm{i}\hbar B)^{1/2}\exp[(\mathrm{i}M/2\hbar B)(Dz^2 - 2zz' + Az'^2)]$ which describes the diffraction spreading of the wave packet and the tidal effects within this wave packet and which is given in terms of the Gaussian optics $ABCD$ matrix coefficients:

$$A = D = \cosh\bigl(\sqrt{\gamma}(t - t_0)\bigr), \qquad B = \frac{1}{\sqrt{\gamma}}\sinh\bigl(\sqrt{\gamma}(t - t_0)\bigr), \qquad C = \gamma B \tag{49}$$

Finally: $L(t) = M(\dot{\xi}^2/2 + \gamma\xi^2/2 - g\xi)$ and the gravitational displacement ξ satisfy $\ddot{\xi} = -g + \gamma\xi$ which yields $\xi = (g/\gamma)[1 - \cosh(\sqrt{\gamma}(t - t_0))]$.

Let us show explicitly the action of this propagator on the lowest-order Gaussian mode:

$$\mathit{wave\_packet}_{@t_0} = \frac{1}{\sqrt{X_0}}\exp\left[\frac{\mathrm{i}M}{2\hbar}\frac{Y_0}{X_0}(z - z_0)^2\right]\exp\bigl[\mathrm{i}Mv_0(z - z_0)/\hbar\bigr] \tag{50}$$

The following result is obtained:

$$
\begin{aligned}
&\mathit{wave\_packet}(z,t)\\
&\quad=\exp\left[\frac{\mathrm{i}M}{\hbar}\dot{\xi}(z-\xi)\right]\exp\left[\frac{\mathrm{i}M}{\hbar}\int_{t_0}^{t}\left(\dot{\xi}^2/2+\gamma\xi^2/2-g\xi\right)\mathrm{d}t_1\right]\\
&\qquad\times\int_{-\infty}^{+\infty}\mathrm{d}z'(M/2\pi\mathrm{i}\hbar B)^{1/2}\exp\left[(\mathrm{i}M/2\hbar B)\left(D(z-\xi)^2-2(z-\xi)z'+Az'^2\right)\right]\frac{1}{\sqrt{X_0}}\\
&\qquad\qquad\times\exp\left[\frac{\mathrm{i}M}{2\hbar}\frac{Y_0}{X_0}(z'-z_0)^2\right]\exp\left[\mathrm{i}Mv_0(z'-z_0)/\hbar\right]\\
&\quad=\exp\left[\frac{\mathrm{i}M}{\hbar}\dot{\xi}(Az_0+Bv_0)\right]\exp\left[-\frac{\mathrm{i}M}{2\hbar}\int_{t_0}^{t}\left(\dot{\xi}^2-\gamma\xi^2\right)\mathrm{d}t_1+\frac{\mathrm{i}M}{\hbar}\dot{\xi}\xi\right]\frac{1}{\sqrt{X}}\\
&\qquad\times\exp\left[\frac{\mathrm{i}M}{2\hbar}\left(ACz_0^2+DBv_0^2+2BCz_0v_0\right)\right]\exp\left[\frac{\mathrm{i}M}{\hbar}(Cz_0+Dv_0+\dot{\xi})(z-\xi-Az_0-Bv_0)\right]\\
&\qquad\times\exp\left[\frac{\mathrm{i}M}{2\hbar}\frac{Y}{X}(z-Az_0-Bv_0-\xi)^2\right]\\
&\quad=\exp\left[\frac{\mathrm{i}S_{\mathrm{cl}}(t,t_0)}{\hbar}\right]\mathit{wave\_packet}_{@t_0}(z-z_{\mathrm{cl}},v_{\mathrm{cl}},X,Y)
\end{aligned}\tag{51}
$$

where

$$
\begin{aligned}
S_{\mathrm{cl}}(t,t_0)&=M\dot{\xi}(Az_0+Bv_0)-\frac{M}{2}\int_{t_0}^{t}\left(\dot{\xi}^2-\gamma\xi^2\right)\mathrm{d}t_1+M\dot{\xi}\xi+\frac{M}{2}\left(ACz_0^2+DBv_0^2+2BCz_0v_0\right)\\
&=\frac{M}{\gamma^{3/2}}\Big[g^2\left(x-\sinh x+\tfrac{1}{2}\sinh 2x\right)+\frac{\gamma}{4}\left(v_0^2-2gz_0+\gamma z_0^2\right)\sinh 2x\\
&\qquad+\sqrt{\gamma}v_0(-g+\gamma z_0)\sinh^2 x\Big]
\end{aligned}\tag{52}
$$

with $x=\sqrt{\gamma}(t-t_0)$, is the classical action and where $\mathit{wave\_packet}_{@t_0}$ is the wave packet at the initial time t_0 in which the central position z_0, the initial velocity v_0 and the initial complex width parameters X_0, Y_0 in phase space have to be replaced by their values at time t given by the A,B,C,D transformation law:

$$
\begin{aligned}
z_{\mathrm{cl}}&=Az_0+Bv_0+\xi, &\qquad X&=AX_0+BY_0\\
v_{\mathrm{cl}}&=Cz_0+Dv_0+\dot{\xi}, &\qquad Y&=CX_0+DY_0
\end{aligned}\tag{53}
$$

In the limit where $\gamma\to 0$, $A=D\to 1$, $B\to t-t_0$, $C\to 0$ and $\xi\to-(1/2)g(t-t_0)^2$ and

$$
S_{\mathrm{cl}}\to\frac{M}{2}(t-t_0)\left(v_0^2-2gz_0-gv_0(t-t_0)+2g^2(t-t_0)^2/3\right)\tag{54}
$$

2.2.1. *The vertical gravimeter/gradiometer*

This corresponds to the experiments of Chu et al. [6,24], in which an atom cloud is launched vertically upwards and submitted to three Raman pulses with time separations T and T', which exchange the momentum $\pm\hbar k/M$ with the atoms. Let us first specify the classical trajectories on both arms of the interferometer i.e. the coordinates of the interaction points and the corresponding velocities (*figure 1*).

The first interaction takes place at z_0 and, when they leave this beam splitter, the atoms have the velocity v_0 on arm I and $v_0+\hbar k/M$ on arm II.

For the second interaction on arm I:

$$
z_1=A(T)z_0+B(T)v_0+\xi(T)\tag{55}
$$

$$
v_1=C(T)z_0+D(T)v_0+\dot{\xi}(T)+\hbar k/M\tag{56}
$$

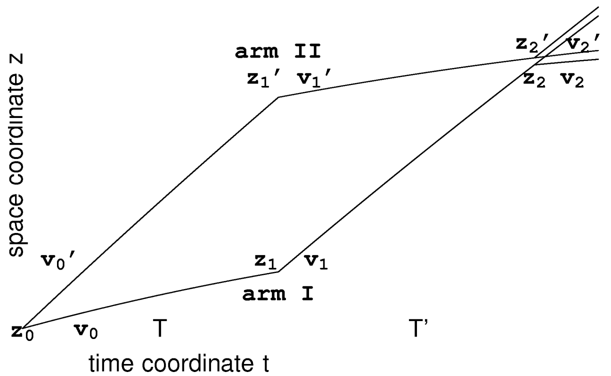

Figure 1. Space–time configuration of an atom wave gravimeter.

and on arm II:

$$z'_1 = A(T)z_0 + B(T)(v_0 + \hbar k/M\ ) + \xi(T) \tag{57}$$

$$v'_1 = C(T)z_0 + D(T)(v_0 + \hbar k/M\ ) + \dot{\xi}(T) - \hbar k/M \tag{58}$$

Finally the third interaction takes place at:

$$z_2 = A(T')z_1 + B(T')v_1 + \xi(T') \tag{59}$$

on arm I and at:

$$z'_2 = A(T')z'_1 + B(T')v'_1 + \xi(T') \tag{60}$$

on arm II, and the respective velocities are:

$$v_2 = C(T+T')\Big(z_0 - \frac{g}{\gamma}\Big) + D(T+T')v_0 + \big[D(T') - 1\big]\frac{\hbar k}{M} \tag{61}$$

$$v'_2 = C(T+T')\Big(z_0 - \frac{g}{\gamma}\Big) + D(T+T')v_0 + \big[D(T+T') - D(T')\big]\frac{\hbar k}{M} \tag{62}$$

These two velocities differ slightly:

$$v_2 - v'_2 = \frac{\hbar k}{M}\big[2\cosh\big(\sqrt{\gamma}T'\big) - \cosh\big(\sqrt{\gamma}(T+T')\big) - 1\big] \tag{63}$$

and therefore, in general, we expect an oscillation with distance in the interference signal as the detector is moved away from the output. There is also a difference in final position of the wave packet centers:

$$z_2 - z'_2 = \frac{\hbar k}{M\sqrt{\gamma}}\big[2\sinh\big(\sqrt{\gamma}T'\big) - \sinh\big(\sqrt{\gamma}(T+T')\big)\big] \tag{64}$$

With the $ABCD$ formalism, it is possible to write explicitly the wave packet expressions along these two paths and to calculate their local amplitude and phase at any point where they overlap at the output of the last field zone, from which one can compute the interference signal by integration over the detection volume. Here, for simplicity, we shall simply consider only their phase difference at a specific point. The classical action is easily calculated along each path with the above formula. A simple Mathematica program gives the final following result:

$$\Delta S = S_1 + S_3 - S_2 - S_4$$

$$= \frac{\hbar k}{\sqrt{\gamma}} \left[2\sinh\left(\sqrt{\gamma}T'\right) - \sinh\left(\sqrt{\gamma}(T+T')\right)\right]$$
$$\times \left[\left(v_0 + \frac{\hbar k}{2M}\right)\cosh\left(\sqrt{\gamma}(T+T')\right) + \sqrt{\gamma}\left(z_0 - \frac{g}{\gamma}\right)\sinh\left(\sqrt{\gamma}(T+T')\right)\right] \quad (65)$$

where $S_1$and S_3 are the contributions from path I and S_2 and S_4 the contributions from path II. We observe that we can rewrite this action as:

$$\Delta S = \left(\frac{z_2 - z_2'}{2}\right)(Mv_2 + \hbar k + Mv_2') \quad (66)$$

so that at the midpoint $z = (z_2 + z_2')/2$, it is exactly cancelled by the propagation phase factor $\exp[\mathrm{i}Mv_{\mathrm{cl}}(z - z_{\mathrm{cl}})/\hbar]$ contained in each wave packet before the last interaction. We see that this action also cancels for the choice of T and T' for which z_2 and z_2' coincide. After the last interaction, there is a small residual term:

$$\pm k\left(\frac{z_2 - z_2'}{2}\right) \quad (67)$$

where the sign depends on the output channel. When this small residual term is added to the laser beam phases, we obtain for the overall phase:

$$\delta\varphi = -k\left((z_2 + z_2')/2 - z_1 - z_1' + z_0\right)$$
$$= -\frac{k}{\sqrt{\gamma}}\left[\left[\sinh\left(\sqrt{\gamma}(T+T')\right) - 2\sinh\left(\sqrt{\gamma}T\right)\right]\left(v_0 + \frac{\hbar k}{2M}\right)\right.$$
$$\left. + \sqrt{\gamma}\left[1 + \cosh\left(\sqrt{\gamma}(T+T')\right) - 2\cosh\left(\sqrt{\gamma}T\right)\right]\left(z_0 - \frac{g}{\gamma}\right)\right] \quad (68)$$

This is an exact result, which can be written to first-order in γ:

$$\delta\varphi = -k\left\{\left[-g\left(\frac{1}{2}(T+T')^2 - T^2\right) - \left(v_0 + \frac{\hbar k}{2M}\right)(T - T')\right]\right.$$
$$+ \gamma\left[-\frac{g}{12}\left(\frac{1}{2}(T+T')^4 - T^4\right) + z_0\left(\frac{1}{2}(T+T')^2 - T^2\right)\right.$$
$$\left.\left. + \left(v_0 + \frac{\hbar k}{2M}\right)\left(\frac{1}{6}(T+T')^3 - \frac{T^3}{3}\right)\right]\right\} \quad (69)$$

If $T = T'$, this reduces to:

$$kgT^2 + k\gamma T^2\left[\frac{7}{12}gT^2 - \left(v_0 + \frac{\hbar k}{2M}\right)T - z_0\right] \quad (70)$$

which is precisely the result used by A. Peters et al. [24], also derived by P. Wolf and Ph. Tourrenc [25] by integration of the Lagrangian over the classical trajectory and which is obtained here directly from the $ABCD$ matrix.

2.2.2. *The cw output from an atom laser*

The results derived above for the propagation of atom waves in gravito-inertial fields can be applied to various experiments in which a coherent atomic beam is extracted vertically from a BEC under the action of gravity in a continuous mode [26,27].

As pointed out in reference [17, p. 353], when $\gamma = 0$, the time-dependent propagator (48):

$$\mathcal{K}(z,z',\tau) = \left(\frac{M}{2\pi\,\mathrm{i}\hbar\tau}\right)^{1/2} \exp\left[\left(\frac{\mathrm{i}M}{2\hbar}\right)\left[-\frac{g^2\tau^3}{12} + g(z+z')\tau + \frac{(z-z')^2}{\tau}\right]\right] \tag{71}$$

may also be derived as the Fourier transform of the time-independent Green function:

$$\widehat{\mathcal{G}}_{\mathrm{E}}(z,z') = \int_0^{+\infty} \mathrm{d}\tau\, \mathrm{e}^{\mathrm{i}E\tau/\hbar}\mathcal{K}(z,z',\tau) = -2\pi\,\mathrm{i}\frac{Ml}{\hbar}\mathrm{Ai}(-\zeta^-)\left[\mathrm{Bi}\left(-\zeta^+\right) + \mathrm{i}\,\mathrm{Ai}\left(-\zeta^+\right)\right] \tag{72}$$

involving the well-known Airy function solutions and where:

$$l = \left(\frac{\hbar^2}{2M^2 g}\right)^{1/3}, \qquad \zeta^{\pm} = \left(\frac{E}{Mgl} + \frac{z+z'}{2l} \pm \frac{|z-z'|}{2l}\right) \tag{73}$$

and, if we assume that the output coupling takes place at a well-defined vertical position z_c, with an output coupling matrix element $\widehat{V}_{12}\delta(z-z_c)$, the integral equation (17) gives to first order in $\widehat{V}_{12}$ (weak-coupling case):

$$\widehat{\Psi}_{\mathrm{E1}}(z) = \frac{1}{\mathrm{i}\hbar}\widehat{\mathcal{G}}_{\mathrm{E}}(z,z_c)\widehat{V}_{12}\,\widehat{\Psi}_{\mathrm{E2}}(z_c) \tag{74}$$

and where $\widehat{\Psi}_{\mathrm{E2}}(z_c)$ is the solution inside the BEC (Thomas–Fermi) at $z = z_c$.

In three dimensions, formulas (71) and (72) become:

$$\mathcal{K}(\vec{r},\vec{r}\,',\tau) = \left(\frac{M}{2\pi\,\mathrm{i}\hbar\tau}\right)^{3/2} \exp\left\{\left(\frac{\mathrm{i}M}{2\hbar}\right)\left[-\frac{g^2\tau^3}{12} + \vec{g}\cdot(\vec{r}+\vec{r}\,')\tau + \frac{|\vec{r}-\vec{r}\,'|^2}{\tau}\right]\right\} \tag{75}$$

and

$$\begin{aligned}\widehat{\mathcal{G}}_{\mathrm{E}}(\vec{r},\vec{r}\,') &= \int_0^{+\infty} \mathrm{d}\tau\, \mathrm{e}^{\mathrm{i}E\tau/\hbar}\mathcal{K}(\vec{r},\vec{r}\,',\tau)\\ &= \frac{M}{2\hbar}\frac{1}{|\vec{r}-\vec{r}\,'|}\left[\mathrm{F}\left(-\rho^+\right)\mathrm{Ai}'(-\rho^-) - \mathrm{F}'\left(-\rho^+\right)\mathrm{Ai}(-\rho^-)\right]\end{aligned} \tag{76}$$

with

$$\mathrm{F}\left(-\rho^+\right) = \mathrm{Ai}\left(-\rho^+\right) - \mathrm{i}\,\mathrm{Bi}\left(-\rho^+\right) \tag{77}$$

and

$$\rho^{\pm} = \left(\frac{E}{Mgl} + \frac{\vec{g}\cdot(\vec{r}+\vec{r}\,')}{2gl} \pm \frac{|\vec{r}-\vec{r}\,'|}{2l}\right) \tag{78}$$

Well-known asymptotic expressions for the Airy functions give simple formulas in the limit of small g. Of course, the overlap integral over the transverse coordinates x', y' in the output plane has to be calculated in this 3D case and it could be a better choice to propagate the 3D wave packets first, using the $ABCD\xi$ law for each mode, and to take the Fourier transform in the end. The reader will find more details on various approximation methods to proceed (low g, paraxial approximation for z, stationary phase point) in references [17,28].

2.2.3. *Stability condition for the gravitational resonator*

A well-known application of the $ABCD$ formalism in Gaussian optics is to allow a discussion of stability conditions of resonators. In the gravitational resonator studied in detail in reference [28], atom waves are reflected from an horizontal mirror with the following $ABCD$ matrix:

$$\begin{pmatrix} 1 & 0 \\ -2v_z/R & 1 \end{pmatrix} \tag{79}$$

where v_z is the atom vertical velocity on the mirror and R the mirror radius. If this is combined with the matrices for free propagation upwards and downwards to generate a consistent round-trip from the mirror surface:

$$M = \begin{pmatrix} 1 & 0 \\ -2v_z/R & 1 \end{pmatrix} \begin{pmatrix} 1 & 2v_z/g \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2v_z/g \\ -2v_z/R & 1 - 4v_z^2/Rg \end{pmatrix} \tag{80}$$

and the stability condition $|\operatorname{Tr}(M)| < 2$ then reads simply $v_z^2 < Rg$ or $h < R/2$, where h is the apex, as discussed in [28]. Clearly, this discussion can be refined with the more general matrices and phase factors introduced above for the wave packets.

Also, using the $ABCD$ law for X, it is easy to show that the focal point for atom waves, lies at $z = R/4$ above the surface of the mirror.

Experiments involving bouncing Bose–Einstein condensates [29] require of course to use the full 3D time-dependent propagators and it will be interesting to make a comparison with purely numerical simulations.

More generally we can expect these $ABCD$ matrices to become an essential tool to calculate complex optical systems in atom optics with sequences of mirrors and lenses especially with magnetic elements, which are easily described with a $\overset{\Rightarrow}{\gamma}$ tensor. It should be emphasized that the formalism can be used in absorbing or amplifying [14–16] media and for any modal structure of the wave packets [17].

2.3. Beam splitter physics and theory

Quite generally a matter-wave beam splitter uses a scattering process such as:

$$A(M_A, E_A, \vec{p}_A) + B(M_B, E_B, \vec{p}_B) \longrightarrow C(M_C, E_C, \vec{p}_C) + D(M_D, E_D, \vec{p}_D) \tag{81}$$

A is the incoming particle to be scattered into the outgoing particle D through the interaction with quasi-particles B and C of the scattering field. As examples let us mention neutron scattering by phonons: $A \equiv D \equiv n$; two-level atom scattering by photons: $A \equiv$ atom in lower state a, $D \equiv$ atom in upper state b, in which case C is present only for two-photon processes (Raman or cascade). A special case is obtained if $a \equiv b$, $B \equiv C$ (Bragg scattering of atoms by a standing laser wave). All these particles may have an effective mass M_X or be massless and their energy E_X is related to their momentum $\vec{p}_X$ through a dispersion law. In free space:

$$E_X(\vec{p}) = \sqrt{M_X^2 c^4 + p_X^2 c^2} \tag{82}$$

We can write a Hamiltonian density for the scattering process, which displays its 4-wave mixing character: $g\phi_D^\dagger(x)\phi_C^\dagger(x)\phi_B(x)\phi_A(x) + \text{h.c.}$ The corresponding S-matrix expresses energy and momentum conservation through Dirac distributions: $S \propto \delta(E_D + E_C - E_B - E_A)\delta(\vec{p}_D + \vec{p}_C - \vec{p}_B - \vec{p}_A)$. To have a well-defined correspondence between the momenta of the incoming and outgoing particles A and D, we must have a well-defined momentum difference for the quasi-particles $\vec{p}_C - \vec{p}_B$ and, to have a well-defined phase between the corresponding fields, the quasi-particles must be in a coherent state corresponding to a coherent effective field: $V_{\text{eff}}(x) \propto \langle \Phi_{BC} | \phi_B(x)\phi_C^+(x) | \Phi_{BC} \rangle + \text{c.c.}$ If $E_C = E_B$, this field is time-independent, the scattering is elastic and the Bragg condition is satisfied. More generally, the masses M_A and M_D may be different, in which case the scattering remains elastic and the Bragg condition satisfied if the resonance condition $\hbar\omega_{\text{eff}} = E_C - E_B = (M_A - M_D)c^2$ is itself satisfied (the direction of the momentum is changed but not its modulus).

To obtain a closed interferometer, a number of scattering zones or events are organized so as to form closed paths in space or in space–time with at least two arms, along which the group velocities may differ transversally, as mentioned before, but also longitudinally. For the dispersion law given above, it is clear that a change in momentum will change the group velocity in the longitudinal (forward) direction for massive particles only. This change of the momentum modulus implies a corresponding change in the kinetic energy which can be obtained only with an off-resonant field (either with a time-dependent plane wave or with a monochromatic localized wave). In this case the velocity change will be proportional to the detuning from resonance. The longitudinal momentum borrowed from the diffracting field must be contained in its Fourier expansion and it is always possible to tilt the field angle to adjust continuously the momentum exchange from purely transverse to longitudinal as in the experiments reported in reference [9].

This velocity change along the forward direction is the basis for the so-called mechanical reinterpretation of Ramsey fringes [4,9,10,20,30]. We will now illustrate this point in more detail through a simple first-order theory of Ramsey fringes.

2.3.1. *First-order theory and reinterpretation of the Ramsey fringes*

Let us consider a beam of two-level atoms with $E_a < E_b$ initially in state a which interacts successively with two field zones respectively centered at x_1 and x_2 and let us calculate the excited state amplitude to first order in each field zone:

$$b^{(1)}(\vec{r},t) = \frac{1}{\mathrm{i}\hbar}\int_{-\infty}^{t}\mathrm{d}t' \int \frac{\mathrm{d}^3p}{(2\pi\hbar)^{3/2}} \int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}} V_{ba}(\vec{k},t')\,\mathrm{e}^{\mathrm{i}\vec{k}\cdot(\vec{r}-\vec{r_1})}\,\mathrm{e}^{\mathrm{i}[E_b(\vec{p}+\hbar\vec{k})-E_a(\vec{p})](t'-t)/\hbar} \times \mathrm{e}^{\mathrm{i}[\vec{p}\cdot\vec{r}-E_a(\vec{p})t]/\hbar}\langle a|\langle\vec{p}\,|\,\Psi^{(0)}\rangle \tag{83}$$

where the energy is given by the dispersion relation (82) and can be expanded in a Taylor series:

$$E(\vec{p}+\hbar\vec{k}) = E(\vec{p}) + \frac{\hbar\vec{k}\cdot\vec{p}c^2}{E(\vec{p})} + \frac{(\hbar k)^2c^2}{2E(\vec{p})} + \cdots = E(\vec{p}) + \hbar\vec{k}\cdot\vec{v} + \hbar\delta + \cdots \tag{84}$$

If the initial wave packet has a narrow width in momentum around $\vec{p}_0$:

$$b^{(1)}_{\vec{p}_0}(\vec{r},t) = \left[\begin{array}{c}\dfrac{1}{\mathrm{i}\hbar}\displaystyle\int_{-\infty}^{t}\mathrm{d}t' \int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}} V_{ba}(\vec{k},t')\,\mathrm{e}^{\mathrm{i}\vec{k}\cdot(\vec{r}-\vec{r_1})} \\ \mathrm{e}^{\mathrm{i}[E_b(\vec{p}_0+\hbar\vec{k})-E_a(\vec{p}_0)](t'-t)/\hbar}\end{array}\right] a^{(0)}_{\vec{p}_0}(\vec{r},t) \tag{85}$$

Let us introduce a monochromatic electromagnetic wave with a Gaussian distribution of k_x [30] and, for simplicity, let us ignore the dimension y:

$$b^{(1)}_{\vec{p}_0}(\vec{r},t) = \mathrm{i}\Omega_{ba}\,\mathrm{e}^{\mathrm{i}(kz-\omega t+\varphi)}\frac{w}{2\sqrt{\pi}}\int_{-\infty}^{+\infty}\mathrm{d}k_x\,\mathrm{e}^{-w^2k_x^2/4}\,\mathrm{e}^{\mathrm{i}k_x(x-x_1)}\,\mathrm{e}^{\mathrm{i}[\omega-\omega_{ba}-kv_z-k_xv_x-\delta](t-t_1)} \times \int_{-\infty}^{t}\mathrm{d}t'\,\mathrm{e}^{-\mathrm{i}[\omega-\omega_{ba}-kv_z-k_xv_x-\delta](t'-t_1)}a^{(0)}_{\vec{p}_0}(\vec{r},t) \tag{86}$$

In the time integral the upper bound t may be extended to infinity, when the considered wave packet has left the interaction zone (this is justified in the footnote,[1] where the exact calculation of reference [30] is recalled). We obtain a δ function expressing energy conservation as expected from the S-matrix: $2\pi\delta(\omega-\omega_{ba}-kv_z-k_xv_x-\delta)$, which annihilates the time precession factor of the pseudo-spin, that we have introduced for the illustration, in formula (86). If we neglect the k_x-dependence of the recoil shift δ, the effect of this energy conservation is to select a particular spatial Fourier component of wave vector

$k_x = (\omega - \omega_{ba} - kv_z - \delta)/v_x$, which clearly appears as a momentum communicated to the atom wave: [2]

$$b^{(1)}_{\vec{p}_0}(\vec{r},t) = \mathrm{i}\frac{\sqrt{\pi}w}{v_x}\Omega_{ba}\,\mathrm{e}^{\mathrm{i}(kz-\omega t+\varphi)}\,\mathrm{e}^{-w^2(\omega-\omega_{ba}-kv_z-\delta)^2/4v_x^2}\,\mathrm{e}^{\mathrm{i}(\omega-\omega_{ba}-kv_z-\delta)(x-x_1)/v_x}a^{(0)}_{\vec{p}_0}(\vec{r},t) \quad (87)$$

hence the Ramsey fringes in the excited state probability:

$$b^{(1)}_{\vec{p}_0}(\vec{r},t)b^{(1)*}_{\vec{p}_0}(\vec{r},t) \propto \mathrm{e}^{\mathrm{i}(\omega-\omega_{ba}-kv_z-\delta)(x_2-x_1)/v_x} + \text{c.c.} \quad (88)$$

In the classical limit, $\hbar \to 0$ for the external motion, the atom may have both a well-defined position and a well-defined velocity:

$$a^{(0)}_{\vec{p}_0}(\vec{r},t)a^{(0)*}_{\vec{p}_0}(\vec{r},t) \propto \delta\big(\vec{r} - \vec{r_0} - \vec{v}(t-t_0)\big) \quad (89)$$

This classical trajectory for a point-like atom introduces a correspondence between space and time and we retrieve the time-dependence of the off-diagonal matrix element $b^{(1)}_{\vec{p}_0}(\vec{r},t)a^{(0)*}_{\vec{p}_0}(\vec{r},t)$ that we have in the usual non-quantized approach. This correspondence is lost if either the atom wave packet spreading or the recoil effects cannot be neglected anymore ($\hbar^2$ terms in the energy expansion). If we write $\vec{p} = \vec{p}_0 + \hbar\overrightarrow{\delta K}$:

$$E\big(\vec{p}_0 + \hbar\overrightarrow{\delta K} + \hbar\vec{k}\big) \simeq E(\vec{p}_0) + \hbar\big(\overrightarrow{\delta K} + \vec{k}\big)\cdot\vec{v} + \frac{\hbar^2k^2}{2M} + \frac{\hbar^2\delta K^2}{2M} + \frac{\hbar^2\overrightarrow{\delta K}\cdot\vec{k}}{2M} + \cdots \quad (90)$$

The last term will introduce corrections which depend on the spatial derivative of the wave packet. The recoil term $\hbar^2k_x^2/2M$ can be kept in the k_x integration in formula (86) and leads to analytical corrections which have been calculated explicitly. They scale with the ratio of the de Broglie wavelength of the atom to the beam waist and lead to significant asymmetries of the lineshape e.g. in the Doppler-free two-photon lineshape of cold hydrogen. They are also expected to play a role in the accuracy of cold atom clocks in the microwaves.

2.3.2. *Strong-field theory of the beam splitters*

In order to describe the propagation of the atom waves in a realistic way inside the beam splitters it is necessary to have an approach valid for strong diffracting fields. Such an approach, presented in references [9,10,19], is the generalization, to a two-level system, of the dynamical diffraction theory valid in the two-beam approximation for neutrons. In the case of a constant scattering field, it turns the time-dependent Schrödinger equation in the rotating frame to a time-independent equation and looks for stationary plane-wave solutions, i.e. corresponding to an energy E and to a momentum $\vec{p} = \hbar\vec{K}$. For each wave of wave vector $\vec{K}$ corresponding to the lower state a, there is a coupled wave with the wave vector $\vec{K} + \vec{k}$ corresponding to the excited state b, and their amplitudes u are coupled by the equations:

$$\left[\frac{\hbar^2}{2M}(\vec{K}+\vec{k})^2 - E + V_{bb} + \hbar(\omega_{ba}-\omega)\right]u_{b,\vec{K}+\vec{k}} - \hbar\Omega_{ba}u_{a,\vec{K}} = 0$$
$$-\hbar\Omega_{ba}u_{b,\vec{K}+\vec{k}} + \left[\frac{\hbar^2}{2M}\vec{K}^2 - E + V_{aa}\right]u_{a,\vec{K}} = 0 \quad (91)$$

Apart from the detuning $(\omega_{ba} - \omega)$ these equations are identical to those which describe neutron interferometry. At resonance or in the special case of Bragg diffraction by a standing laser wave ($\omega_{ba} = \omega_{\text{eff}} = 0$), the physics is expected to be exactly the same. The internal label a or b is simply added to the external one $\vec{K}$ or $\vec{K} + \vec{k}$. From the compatibility condition of the coupled equations (91) one obtains the dispersion relation for the waves that may propagate in this special crystal made up of light:

$$\left[\frac{\hbar^2}{2M}\big(\vec{K}+\vec{k}\big)^2 - E + V_{bb} + \hbar(\omega_{ba}-\omega)\right]\left[\frac{\hbar^2}{2M}\vec{K}^2 - E + V_{aa}\right] - \hbar^2\Omega_{ba}^2 = 0 \quad (92)$$

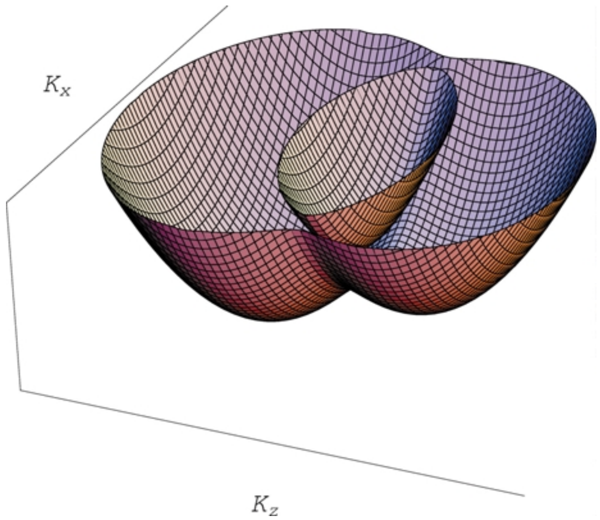

Figure 2. The dispersion surface for atom waves in the beam splitter out-of-resonance. The energy E is displayed as a function of the normal and tangential components of the atom wave vector.

The corresponding dispersion surface is displayed in *figure 2*, with the same parameters as in reference [19].

From the cut by a constant energy plane, one can show that eight waves may propagate in this crystal, four of which are reflected waves and four propagate in the forward direction. They group themselves in two pairs which differ by the momentum $\hbar\vec{k}$ and by their internal state. Within each pair the two solutions (corresponding to the α and β branches of neutron dynamical diffraction theory) differ by a small momentum difference in the forward direction. These waves beat together, producing what is called *pendellösung* oscillations for neutron waves [31] and Rabi oscillations in atomic physics [32]. As in the case of Ramsey fringes, the quantization of the external motion gives a new picture for the Rabi oscillations, common to a particle without internal structure and to two-level atomic systems.

The dispersion surface (92) can also be used to discuss the group velocity properties within the splitter. The atomic currents exhibit a very peculiar behaviour discussed in references [10,19] and lead to the so-called Borrmann effect. The detailed knowledge of these propagation properties (directions of the current flows, effective masses, etc.) within the splitter is essential to be able to calculate properly the phase shifts due to other fields, in actual interferometers.

The time-dependent approach [20] to beam splitters in the strong-field regime follows from equation (12) for the $\widetilde{S}$-matrix, which leads also, out-of-resonance, to a momentum increase/decrease proportional to the detuning in the forward direction. When combined with the propagator (48) this $\widetilde{S}$-matrix, used once on the way up and a second time on the way down, is the basis for the theory of atomic fountain clocks.

2.3.3. *Influence of gravitation on the beam splitting process*

In precision experiments with matter-wave interferometers, one cannot ignore the fact that gravitation also acts on the atom waves within the beam splitter and this was already found to be critical in neutron interferometry. The ideal way to handle this problem is to look for an exact solution of Schrödinger equation with all fields included in the Hamiltonian. Such exact solutions have been found in the case of square temporal laser pulses in the presence of a simple acceleration term accounting for the earth gravity [33,34]. In other cases only approximate solutions such as WKB solutions can be used to derive the phase shifts [35]. One could also use the $ABCD\xi$ formalism in a fictitious effective medium, which has the dispersion properties of the laser beam splitters.

3. General relativistic framework for atom interferometry [10,12,13]

It is possible to include all possible effects of inertial fields, as well as all the general relativistic effects of gravitation in a consistent and synthetic framework, in which the atomic fields are second-quantized. The starting point is the use of coupled field equations for atomic fields of a given spin in curved space–time: e.g., coupled Klein–Gordon, Dirac or Proca equations. Gravitation is described by the metric tensor $g_{\mu\nu}$ and by tetrads, which enter in these equations. Several strategies can then be adopted: one can perform Foldy–Wouthuysen transformations [36], but conceptual difficulties arise in the case of arbitrary $g_{\mu\nu}$; one can go to the weak-field limit $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$ and use renormalized spinors and finally one can consider $h_{\mu\nu}$ as a spin-two tensor-field in flat space–time [37–40] and use ordinary relativistic quantum field theory. Using this last approach, it has been possible to derive field equations that display all interesting terms, coupling Dirac atomic fields, gravitational and electromagnetic fields and simple expressions of the corresponding relativistic phase shifts in atom interferometers [13].

The evolution equation of the state vector $|\Psi(t)\rangle$ in the interaction picture is:

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\big|\Psi(t)\big\rangle = \int \mathrm{d}^3x\,\theta^\dagger(x)\mathcal{V}_G(x)\theta(x)\big|\Psi(t)\big\rangle \tag{93}$$

where the operator $\mathcal{V}_G(x)$, acting on the field operator $\theta(x)$, is given in compact form by:

$$\mathcal{V}_G = \frac{c}{4}\alpha^\mu h_{\mu\nu}p^\nu + \text{h.c.} = \frac{c}{4}\big\{\alpha^\mu h_{\mu\nu}, p^\nu\big\}_+ \quad \text{with} \tag{94}$$

$$p^0 = -\alpha^j p_j + \gamma^0 mc \quad \text{and} \quad p_j = \mathrm{i}\hbar\partial_j \tag{95}$$

The free field operator θ is written as:

$$\theta(x) = \sum_{r=1}^{2}\int \mathrm{d}^3p\big[c_r(\vec{p})\chi^{(+)}_{\vec{p},r}(x) + d_r^\dagger(\vec{p})\chi^{(-)}_{\vec{p},r}(x)\big]$$

where $c_r(\vec{p})$ and $d_r(\vec{p})$ are the annihilation operators for the particles or antiparticles, respectively, and $\chi^{(\pm)}_{\vec{p},r}$ are the positive or negative energy solutions of the free Dirac equation [41–43]:

$$\chi^{(\pm)}_{\vec{p},r}(x) = \frac{1}{(2\pi\hbar)^{3/2}}\sqrt{\frac{Mc^2}{E(\vec{p})}}\,u^{(r)}_{(\pm)}(\vec{p})\,\mathrm{e}^{\mp\mathrm{i}(E(\vec{p})t-\vec{p}\cdot\vec{r})/\hbar} \tag{96}$$

We are interested in the output spinor corresponding to one-particle (antiparticle) states: e.g. $\psi(x) = \langle 0|\theta(x)|\Psi(t)\rangle$ for atoms. The evolution of this spinor is governed by the equation:

$$\mathrm{i}\hbar\partial_t\psi = -\mathrm{i}\hbar c\gamma^0\gamma^j\partial_j\psi + Mc^2\gamma^0\psi + \mathcal{V}_G(x)\psi \tag{97}$$

to which we may add terms corresponding to diagonal magnetic dipole and off-diagonal electric dipole interactions [10,12]. This equation has been used in reference [10,13] to discuss all the terms that lead to a phase shift in an interferometer.

For the phase shift, the general result is:

$$\begin{aligned}\delta\varphi = -\frac{1}{\hbar}\int_{t_0}^{t}\mathrm{d}t'\bigg\{&\frac{c^2}{2E(\vec{p})}p^\mu h_{\mu\nu}(\vec{x}_0+\vec{v}t',t')p^\nu + \frac{\gamma}{m(\gamma+1)}\bigg[\frac{c^2p^\mu\vec{\nabla}h_{\mu\nu}(\vec{x}_0+\vec{v}t',t')p^\nu}{2E^2(\vec{p})}\times\vec{p}\bigg]\cdot\vec{s} \\ &- \frac{c}{2}\bigg[\vec{\nabla}\times\bigg(\vec{h}(\vec{x}_0+\vec{v}t',t') - \vec{\vec{h}}(\vec{x}_0+\vec{v}t',t')\cdot\frac{\vec{p}c}{E(\vec{p})}\bigg)\bigg]\cdot\vec{s}\bigg\}\end{aligned} \tag{98}$$

Table 1. Classification of the various energy terms entering the expression of the phase shift. The factor γ is the time dilation factor and should not be confused with the PPN parameter γ_{PPN} which can also be introduced (for de Sitter precession it gives the familiar factor $(\gamma_{\mathrm{PPN}}+1/2)$)

Corresponding energy term V	$h_{\mu\nu}$	Name of the effect
$Eh_{00}/2$	Newtonian potential: $h_{00}=2U/c^2=-2\vec{g}\cdot\vec{x}/c^2$	Gravitational red shift
	or acceleration field $h_{00}=2\vec{a}\cdot\vec{x}/c^2$	Acceleration shift
	Gravity gradient $\vec{g}(z)\cdot\vec{x}=-(g-g'z/2)z$	
	or curvature $R_{0i0j}x^ix^j$	
	Fermi gauge: $h_{00}^F=\ddot{h}_+(t-z/c)\cdot(x^2-y^2)/2$	Effect of
	$+\ddot{h}_\times(t-z/c)\cdot xy$	gravitational waves
$\frac{\gamma}{2m(\gamma+1)}(\vec{\nabla}h_{00}\times\vec{p})\cdot\bar{\vec{s}}$	$h_{00}=2U/c^2$ gives $V=\frac{1}{mc^2}\frac{\gamma}{\gamma+1}[\vec{\nabla}U\times\vec{p}]\cdot\bar{\vec{s}}$	Thomas precession
$-c\vec{p}\cdot\vec{h}$	Rotating frame: $\vec{h}=\vec{\Omega}\times\vec{x}/c$	Sagnac effect
	gives $V=-\vec{\Omega}\cdot\vec{L}$	
	h_{0i} given by the Lense–Thirring metric	Lense–Thirring (orbital)
$-(c/2)[\vec{\nabla}\times\vec{h}]\cdot\bar{\vec{s}}$	$\vec{h}=\vec{\Omega}\times\vec{x}/c$ gives $V=-\vec{\Omega}\cdot\bar{\vec{s}}$	Spin-rotation interaction
	h_{0i} given by the Lense–Thirring metric	Lense–Thirring (spin)
$-\frac{\gamma}{m(\gamma+1)}\left[\vec{\nabla}(c\vec{p}\cdot\vec{h}/E)\times\vec{p}\right]\cdot\bar{\vec{s}}$		$\sim$ Thomas for rotation
$c^2\vec{p}\cdot\overset{\Rightarrow}{h}\cdot\vec{p}/2E$	Schwarzschild metric in isotropic coordinates:	
	$h_{00}=h_{11}=h_{22}=h_{33}=2U/c^2$ gives	
	$V=p^2U/E$ in addition to EU/c^2 from h_{00}	
	Einstein gauge: $h_{11}=-h_{22}=h_+(t-z/c)$,	Effect of gravitational
	$h_{12}=h_{21}=h_\times(t-z/c)$	waves
$(c/2)[\vec{\nabla}\times(\overset{\Rightarrow}{h}\cdot\vec{p}c/E)]\cdot\bar{\vec{s}}$	Schwarzschild metric: $U=-GM/r$	
	$h_{00}=h_{11}=h_{22}=h_{33}=2U/c^2$	de Sitter
	gives $V=\frac{1}{mc^2}\frac{1}{\gamma}[\vec{\nabla}U\times\vec{p}]\cdot\bar{\vec{s}}$ in addition	or geodetic precession
	to $V=\frac{1}{mc^2}\frac{\gamma}{\gamma+1}[\vec{\nabla}U\times\vec{p}]\cdot\bar{\vec{s}}$ from h_{00}	
	Einstein gauge: h_{ij}	Interaction of the spin
		with gravitational waves
$\frac{\gamma}{2m(\gamma+1)}[\vec{\nabla}(c^2\vec{p}\cdot\overset{\Rightarrow}{h}\cdot\vec{p}/E^2)\times\vec{p}]\cdot\bar{\vec{s}}$		$\sim$ Thomas for gravitation

where $\bar{\vec{s}}$ is the mean spin vector:

$$\bar{\vec{s}}=\sum_{r,r'}\beta^*_{r,i}\beta_{r',i}\hbar w^{(r)\dagger}\vec{a}w^{(r')}/2\gamma \tag{99}$$

where $\vec{a}=(\vec{\sigma}_\perp+\gamma\vec{\sigma}_\parallel)$ is the spatial part of the Thomas–Pauli–Lubanski 4-vector operator [44,45].

Expression (98) displays all the terms which may lead to a gravitational phase shift in a matter-wave interferometer. They are summarized in *table 1* where one finds successively:

- the terms involving h_{00} lead to the gravitational shift ($h_{00} = -2\ \vec{g} \cdot \vec{r}/c^2$), to shifts involving higher derivatives of the gravitational potential and to the analog of the Thomas precession (spin-orbit coupling corrected by the Thomas factor);
- the terms which involve $\vec{h} = \{h^{0k}\}$, give the Sagnac effect in a rotating frame ($\vec{h} = \vec{\Omega} \times \boldsymbol{r}/c$), the spin-rotation coupling and a relativistic correction (analogous to the Thomas term for h_{00}). They describe also the Lense–Thirring effects coming from inertial frame-dragging by a massive rotating body, which is a source for $\vec{h}$;
- the other terms, which involve the tensor $\overset{\Rightarrow}{h} = \{h^{ij}\}$ describe genuine general relativity effects such as the effect of gravitational waves and de Sitter geodetic precession (which also includes the Thomas term for $h_{00}{}^3$).

In fact the phase calculation is usually more involved since formula (98) applies only to the case of straight unperturbed trajectories. In practice, however, one cannot ignore the fact that, when calculating the phase to first-order for a given term of the Hamiltonian, the motion of the particles is affected by other terms. One example, mentioned above, is the calculation of the gravitational shift within the atom beam splitters, in which one cannot ignore the important effects of the diffracting electromagnetic field on the trajectories of the particles [10,19,33,35]. Gravitational phase shifts have to be calculated along these trajectories. Another example is the gravity field itself, which, on earth, gives parabolic trajectories for atoms. The phase shift for the other terms in equation (98) has to be calculated along these parabolas. A convenient way to achieve these calculations is to replace $\vec{x}_0 + \vec{v}t'$ and $\vec{v}$ in equation (98) by the classical trajectory $\{\vec{x}(t'), \vec{v}(t')\}$ obtained in the $ABCD$ formalism. In the non-relativistic limit, one is brought back to the Schrödinger equation and, up to second degree in position and momentum operators, the best approach is to take the full benefit of the $ABCD$ formalism developed above, which gives exact results. Higher-order terms can be treated as perturbations along unperturbed trajectories.

The reader will find calculations of the phases corresponding to the various terms in references [3,7,10, 24,25,46–48]. In these calculations, one should never forget that the external field $h_{\mu\nu}$ acts not only on the atoms but also on other components of the experiments, such as mirrors and laser beams and that, depending on the chosen gauge, additional contributions may enter in the final expression of the phase which should, of course, be gauge independent. As an example, the Sagnac phase which can be removed from the above formula by a simple coordinate transformation will reappear in the beam splitter phases.

The expressions valid for spins 0 and $1/2$ may be conjectured to be valid for arbitrary spin if $\vec{\sigma}/2$ is replaced by the corresponding spin operator $\vec{S}$. The extension of the formulas is presently underway using higher spin formalisms.

Formula (98) also displays the analogy with electromagnetic interactions: $h_{\mu\nu}p^\nu/2$ plays the role of the 4-potential A_μ and $(E(\vec{p})/2c)\vec{\nabla} \times (\vec{h} - \overset{\Rightarrow}{h} \cdot \vec{p}c/E(\vec{p}))$ plays the role of a gravitomagnetic field $\vec{\nabla} \times \vec{A}$. This new correspondence [49] between the gravitational interaction and the electromagnetic interaction generalizes the so-called gravitoelectric and gravitomagnetic interactions introduced by de Witt [50] and Papini [51].

The Linet–Tourrenc [52] term $(c^2/2E(\vec{p}))p_\mu h^{\mu\nu}p_\nu$, which appears also in the generalized Thomas precession corresponds to $u_\mu A^\mu/\gamma$ where u_μ is the 4-velocity p_μ/M and the corresponding circulation of potential takes the form of the Aharonov–Bohm phase formula $\oint A^\mu\, \mathrm{d}x_\mu$. Using Stokes theorem in four dimensions [53], this integral gives the phase shift as the ratio of the flux of gravitoelectromagnetic forces through the interferometer space or spacetime area to a quantum of flux of force $\hbar$ or $\hbar c$.

In present earth gravity measurements, the relative sensitivity is $\delta g/g \simeq 3 \cdot 10^{-9}$ after 60 seconds and the absolute accuracy $5 \cdot 10^{-9}$ [6,24]. For rotations, the best sensitivity achieved up to now is $6 \cdot 10^{-10}$ rad·s^{-1}·Hz$^{-1/2}$ [54] but these numbers are expected to improve rapidly in the near future, especially in space experiments, in which general relativistic effects should become detectable. An accurate measurement of the effect of gravitation and inertia on antimatter also appears as a possibility already discussed in reference [55] with a transmission-grating interferometer, although we believe, for obvious reasons, that an antiatom interferometer using laser beams for the antihydrogen beam splitters (so-called Ramsey–Bordé interferometers) would be better suited for such an experiment. The formalism introduced to deal with antiatoms should be useful to discuss such experiments, especially when coherent beams of antihydrogen will be produced either by Bose–Einstein condensation and/or by stimulated bosonic amplification [4] [14–16].

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[1] The exact calculation gives:

$$b^{(1)}_{\overrightarrow{p}0}(\overrightarrow{r},t) = \mathrm{i}\Omega_{ba}\,\mathrm{e}^{\mathrm{i}(kz-\omega t+\varphi)}\int_0^{+\infty}\mathrm{d}\tau\,\mathrm{e}^{-(x-v_x\tau-x_1)^2/w_0^2}\,\mathrm{e}^{[\mathrm{i}(\omega-\omega_{ba}-kv_z-\delta)-\gamma]\tau}a^{(0)}_{\overrightarrow{p}0}(\overrightarrow{r},t)$$
$$= \mathrm{i}\frac{\sqrt{\pi}w}{2v_x}\Omega_{ba}\,\mathrm{e}^{\mathrm{i}(kz-\omega t+\varphi)}\,\mathrm{e}^{-(x-x_1)^2/w_0^2}w(\mathrm{i}\zeta)a^{(0)}_{\overrightarrow{p}0}(\overrightarrow{r},t) \tag{100}$$

with $\zeta = -(x-x_1)/w_0 + [\gamma - \mathrm{i}(\omega-\omega_{ba}-kv_z-\delta)]w_0/2v_x$ and where $w(z)$ is the error function of complex arguments. Using $w(\mathrm{i}\zeta) = 2\exp(\zeta^2) - w(-\mathrm{i}\zeta)$ and in the limit $\gamma \to 0$, we obtain the excited state wave packet:

$$b^{(1)}_{\overrightarrow{p}0}(\overrightarrow{r},t) = \mathrm{i}\frac{\sqrt{\pi}w}{2v_x}\Omega_{ba}\,\mathrm{e}^{\mathrm{i}(kz-\omega t+\varphi)}\left[2\,\mathrm{e}^{-w^2(\omega-\omega_{ba}-kv_z-\delta)^2/4v_x^2}\,\mathrm{e}^{\mathrm{i}(\omega-\omega_{ba}-kv_z-\delta)(x-x_1)/v_x} - w(-\mathrm{i}\zeta)\right]$$
$$\times\, a^{(0)}_{\overrightarrow{p}0}(\overrightarrow{r},t) \tag{101}$$

where the second term vanishes with the distance $(x-x_1)/w_0$ leaving the accelerated or decelerated first contribution as the dominant one.

[2] If, instead of a monochromatic non-plane (localized) wave, we had considered a plane non-monochromatic (pulsed in the time domain) wave, the exchange in momentum would have been replaced by an exchange of energy and the space-dependent phase by a time-dependent phase.

[3] Some authors reserve the name ‘Thomas precession’ for the contribution coming specifically from an acceleration $\vec{a}$ (which has been included here in h_{00}) and separate it from de Sitter precession.

[4] Using, for example, a reaction like: antiproton + positronium → antihydrogen + electron.

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