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Quantum theory of atomic clocks and gravito-inertial sensors: an update

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Abstract
In the framework of the *ABCDξ* formulation of atom optics and with an adequate modelization of the beam splitters, we establish an exact analytical phase shift expression for atom interferometers. This result is valid for a time-dependent external Hamiltonian at most quadratic in position and momentum operators and is expressed in terms of coordinates and momenta of the wave packet centres at the interaction vertices only. As a specific application, the case of atom gyrometers and accelerometers is presented in detail.

Keywords: Atom optics, atom interferometer, inertial sensor, atomic clock, wave packet, beam splitter, *ABCD* matrices, phase shift

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Recently atom interferometers [1] have been described by the *ABCDξ* formalism of atom optics [2–4] which is particularly well suited for the treatment of atomic wave packets. The *ABCD* method in atom optics is inspired by the corresponding method in usual optics. A detailed tutorial comparison of its applications to both fields was given by one of the present authors in Les Houches lectures [4] in 1990, where this formalism was first introduced for matter waves. Further developments and applications of this method can be found in [2] and [3].

This framework incorporates the two main parts of an interferometer: the propagation of wave packets between two beam splitters and the beam splitters themselves. The first stage is achieved through the *ABCDξ theorem* [2] which gives the evolution of a general wave packet in the case of a time dependent external Hamiltonian at most quadratic in position and momentum operators. The second problem is addressed by the *ttt theorem* [5] which specifies the effect of a beam splitter, when its dispersive nature is neglected.

For essentially all applications of atom interferometry today all the physics is well described by the at most quadratic Hamiltonian (gravity, rotation, field gradient, . . .). In any case, if higher terms are added, the problem cannot be solved exactly and a perturbation approach has to be used. Clearly, one should start with the exact solution corresponding to the at most quadratic case. Furthermore, this framework is three dimensional with time-dependent terms which may mix the three dimensions in the course of time (non-orthogonal atom optics) and no assumption is made with respect to the shape of the wave packets. Any wave packet can be expanded, for example, on the complete basis of 3D Hermite–Gauss functions (see [3]) and the theorems used in this paper apply to this general case.

After a brief summary of the framework in section 2, we focus on the two theorems established in [6] (the four-end-point theorem and the phase shift formula). The first one introduces the idea of homologous paths and gives an expression of the classical action variation in terms of the coordinates and momenta of the four end-points only. As any interferometer geometry can be sliced into pairs of such homologous paths we can apply this basic tool to each pair, and thus obtain a compact expression for the phase shift in the case of an arbitrary beam-splitter configuration and for such a Hamiltonian.

Then we discuss several particular cases: identical masses, symmetrical geometry, phase shift after the output spatial integration, time-independent Hamiltonians. As an illustrative application, we treat in detail the symmetrical Ramsey–Bordé interferometer when it is rotating either with Earth (such as the GOM developed in Paris or the CASI developed in Hanover) or fixed on a rotating vehicle (such as satellites e.g. HYPER [7]), and when a gravitational field (gravity + gradient) acts on the atoms. Other applications include the gravimeters and gradiometers developed at Yale and Stanford Universities (groups of Chu [8] and Kasevich [9]). Finally we outline that the phase shift formula can be used to treat atomic clocks.

2. The $ABCD\xi$ framework

In this framework we consider a Hamiltonian which is the sum of an internal Hamiltonian H_0 (with eigenvalues written with rest masses m_i) and of an external Hamiltonian H_{ext}:

$$H_{ext} = \frac{1}{2m}\vec{p}_{op}\cdot\overset{\Rightarrow}{\beta}(t)\cdot\vec{p}_{op} - \frac{m}{2}\vec{q}_{op}\cdot\overset{\Rightarrow}{\gamma}(t)\cdot\vec{q}_{op} - \vec{\Omega}(t)\cdot(\vec{q}_{op}\times\vec{p}_{op}) - m\vec{g}(t)\cdot\vec{q}_{op} \qquad (1)$$

where one recognizes several usual gravito-inertial effects: rotation in $\vec{\Omega}(t)$, gravity in $\vec{g}(t)$, gradient of gravity in $\overset{\Rightarrow}{\gamma}(t), \ldots$ and where $\overset{\Rightarrow}{\beta}(t)$ is usually taken equal to the unity tensor in the absence of gravitational waves.

2.1. The ABCDξ theorem

For the temporal evolution of a general wave packet $\psi(q,t_1) = wp(t_1, q-q_1, p_1, X_1, Y_1)$, where q_1 is the initial central position of the wave packet, p_1 its initial central momentum and (X_1, Y_1) its initial complex width parameters in phase space, one has the $ABCD\xi$ theorem [2]:

$$\psi(q,t_2) = e^{\frac{i}{\hbar}S_{cl}(t_2,t_1,q_1,p_1)}wp(t_2, q-q_2, p_2, X_2, Y_2) \qquad (2)$$

where q_2, p_2, X_2, Y_2 obey the $ABCD$ law (β and $\mathcal{R}$ are the representative matrices of $\overset{\Rightarrow}{\beta}(t)$ and of the rotation operator, and we write A_{21} instead of $A(t_2,t_1)$ for simplicity):

$$\begin{pmatrix} q_2 \\ p_2/m \end{pmatrix} = \begin{pmatrix} \mathcal{R}_{21}\xi_{21} \\ \beta_2^{-1}\mathcal{R}_{21}\dot{\xi}_{21} \end{pmatrix} + \begin{pmatrix} A_{21} & B_{21} \\ C_{21} & D_{21} \end{pmatrix}\begin{pmatrix} q_1 \\ p_1/m \end{pmatrix} \qquad (3)$$

$$\begin{pmatrix} X_2 \\ Y_2 \end{pmatrix} = \begin{pmatrix} A_{21} & B_{21} \\ C_{21} & D_{21} \end{pmatrix}\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} \qquad (4)$$

where S_{cl} is the classical action, and where ξ is the part of q_2 which depends on $\vec{g}(t)$, written here in the non-rotating frame.

For example, the phase of Hermite–Gauss wave packets is (for simplicity we shall omit the transposition sign $\sim$ on matrix representations of vectors)

$$S_{cl}(t_2,t_1,q_1,p_1)/\hbar + p_2(q-q_2)/\hbar + \frac{m}{2\hbar}(q-q_2)\,\mathrm{Re}(Y_2X_2^{-1})(q-q_2) \qquad (5)$$

and, in this case, the main part of the phase shift between t_1 and t_2 is equal to

$$S_{cl}(t_2,t_1,q_1,p_1)/\hbar + p_1q_1/\hbar - p_2q_2/\hbar. \qquad (6)$$

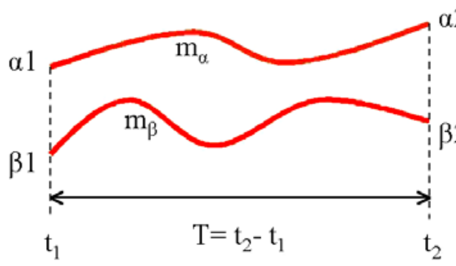

Figure 1. A pair of homologous paths.

2.2. The ttt theorem

When the dispersive properties of a laser beam splitter are neglected (i.e. the wave packet shape is preserved), its effect may be summarized by the introduction of both a phase and an amplitude factor (see [5] for a detailed proof, and [10]):

$$M_{ba}e^{-i(\omega^*t^*-k^*q^*+\varphi^*)} \qquad (7)$$

where t^* and q^* depend on t_A and q_A, the central time and position of the electromagnetic wave used as a beam splitter.

For a temporal beam splitter, the parameters of this modelization are

$$\begin{aligned} t^* &= t_A \\ q^* &= q_{cl}(t_A) \\ k^* &= k \\ \omega^* &= \omega \\ \varphi^* &= \varphi \text{ (laser phase)} \end{aligned} \qquad (8)$$

and for a spatial beam splitter

$$\begin{aligned} q^* &= q_A \\ t^* &\text{ such that } q_{cl}(t^*) = q_A \\ k^* &= k + \delta k \\ \omega^* &= \omega \\ \varphi^* &= \varphi + \delta\varphi \end{aligned} \qquad (9)$$

where q_{cl} is the central position of the input atomic wave packet (equal to the classical position owing to Ehrenfest's theorem), where δk is the additional momentum transferred to the excited atoms out of resonance and where $\delta\varphi$ is a laser phase: $\delta\varphi := -\delta k q_A$ (see [5]). Let us emphasize that these calculations do not rely on the assumption that the splitter is infinitely thin or that the atom trajectories are classical.

3. The four-end-point theorem for a Hamiltonian at most quadratic in position and momentum operators

We shall cut any interferometer into as many slices as there are interactions on either arm and thus obtain several pieces of path which have a common drift time (see section 4). From now on, we shall consider systematically pairs of these homologous paths (see figure 1) in the case of a Hamiltonian at most quadratic.

These two classical trajectories are labelled by their corresponding mass (m_α and m_β), their initial position and

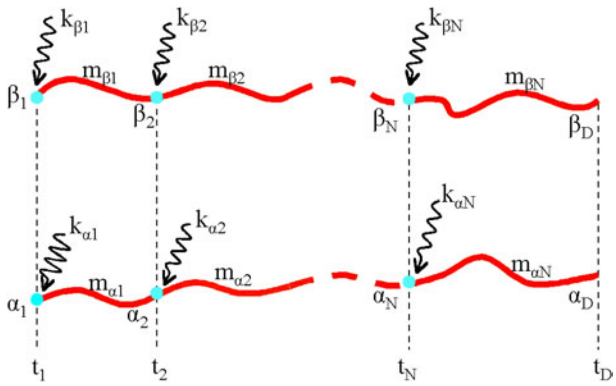

Figure 2. Interferometer geometry sliced into pairs of homologous paths between interactions on either arm (when an interaction only occurs on one arm the corresponding k on the other arm is set $= 0$).

momentum ($q_{\alpha1}$, $p_{\alpha1}$, $q_{\beta1}$ and $p_{\beta1}$) and their common drift time $T = t_2 - t_1$.

If we compare the classical actions along these two paths we obtain the following result (see [6] for a detailed proof):

Theorem 1.

$$\begin{aligned}&\frac{S_{cl}(t_2,t_1,q_{\alpha1},p_{\alpha1})}{m_\alpha} - \frac{p_{\alpha2}}{m_\alpha}q_{\alpha2} + \frac{p_{\alpha1}}{m_\alpha}q_{\alpha1}\\ &\quad - \left[\frac{S_{cl}(t_2,t_1,q_{\beta1},p_{\beta1})}{m_\beta} - \frac{p_{\beta2}}{m_\beta}q_{\beta2} + \frac{p_{\beta1}}{m_\beta}q_{\beta1}\right]\\ &= \left(\frac{p_{\beta2}}{m_\beta} - \frac{p_{\alpha2}}{m_\alpha}\right)\left(\frac{q_{\alpha2}+q_{\beta2}}{2}\right)\\ &\quad - \left(\frac{p_{\beta1}}{m_\beta} - \frac{p_{\alpha1}}{m_\alpha}\right)\left(\frac{q_{\alpha1}+q_{\beta1}}{2}\right) \qquad (10)\end{aligned}$$

which depends on the coordinates and the momenta of the four end-points only.

Actually, as the left-hand side gives the plane-wave part of the phase shift between the two paths, this basic theorem gives the main part of the phase shift expressed with the half sums of the coordinates (mid-points) and the momenta of the four end-points only.

In the case of identical masses ($m_\alpha = m_\beta$) this expression simplifies to

$$\begin{aligned}&S_{cl}(t_2,t_1,q_{\alpha1},p_{\alpha1}) - p_{\alpha2}q_{\alpha2} + p_{\alpha1}q_{\alpha1}\\ &\quad - [S_{cl}(t_2,t_1,q_{\beta1},p_{\beta1}) - p_{\beta2}q_{\beta2} + p_{\beta1}q_{\beta1}]\\ &= (p_{\beta2}-p_{\alpha2})\left(\frac{q_{\alpha2}+q_{\beta2}}{2}\right) - (p_{\beta1}-p_{\alpha1})\left(\frac{q_{\alpha1}+q_{\beta1}}{2}\right). \qquad (11)\end{aligned}$$

4. The phase shift formula for a Hamiltonian at most quadratic in position and momentum operators

For a sequence of pairs of homologous paths (an interferometer geometry) (see figure 2) one can infer the general sum for the main coordinate dependent part of the global phase shift [6]:

$$\begin{aligned}&\frac{p_{\beta D}-p_{\alpha D}}{\hbar}\left(q - \frac{q_{\alpha D}+q_{\beta D}}{2}\right) - \frac{p_{\alpha1}+p_{\beta1}}{2\hbar}(q_{\beta1}-q_{\alpha1})\\ &\quad + \sum_{i=1}^{N}(k_{\beta i}-k_{\alpha i})\frac{q_{\alpha i}+q_{\beta i}}{2}. \qquad (12)\end{aligned}$$

If we now take into account the other terms of the phase shift, we finally obtain the following result [6] (given here for a Gaussian wave packet):

Theorem 2.

$$\begin{aligned}&\Delta\phi(q, t_{N+1}=t_D) = (p_{\beta D}-p_{\alpha D})\left(q - \frac{q_{\alpha D}+q_{\beta D}}{2}\right)\Big/\hbar\\ &\quad - \frac{p_{\alpha1}+p_{\beta1}}{2\hbar}(q_{\beta1}-q_{\alpha1})\\ &\quad + \sum_{i=1}^{N}\left[(k_{\beta i}-k_{\alpha i})\frac{q_{\alpha i}+q_{\beta i}}{2} - (\omega_{\beta i}-\omega_{\alpha i})t_i + \varphi_{\beta i} - \varphi_{\alpha i}\right]\\ &\quad + \sum_{i=1}^{N}\left(\frac{m_{\beta i}}{m_{\alpha i}} - 1\right)\left[\frac{S_{\alpha i}}{\hbar} + \frac{p_{\alpha,i+1}}{2\hbar}(q_{\beta,i+1}-q_{\alpha,i+1})\right.\\ &\quad \left. - \frac{p_{\alpha i}+\hbar k_{\alpha i}}{2\hbar}(q_{\beta i}-q_{\alpha i})\right]\\ &\quad + \frac{m_{\beta,N}}{2\hbar}(q-q_{\beta D})\,\mathrm{Re}(Y_D X_D^{-1})(q-q_{\beta D})\\ &\quad - \frac{m_{\alpha,N}}{2\hbar}(q-q_{\alpha D})\,\mathrm{Re}(Y_D X_D^{-1})(q-q_{\alpha D}) \qquad (13)\end{aligned}$$

where $S_{\alpha i} := S_{cl}(t_{i+1}, t_i, q_{\alpha i}, p_{\alpha i} + \hbar k_{\alpha i}, m_{\alpha i})$.

This basic formula is valid for a time-dependent Hamiltonian and takes into account all the mass differences which may occur. It allows us to calculate exactly the phase shift for all the interferometer geometries which can be sliced as above: symmetrical Ramsey–Bordé (Mach–Zehnder), atomic fountain clocks, ...

Let us point out that the nature (temporal or spatial) of beam splitters leads to different slicing. In the spatial case, indeed, the number of different t_i^* may be twice as great as in the temporal case (see the definition of t_i^* in these two different cases in 2.2).

5. Particular cases

5.1. Phase shift after spatial integration

In any interferometer one has to integrate spatially the output wave packet over the detection region. For example, with Gaussian wave packets this integration leads to a mid-point theorem [3]: *the first term of $\Delta\phi(q, t_D)$ disappears when the spatial integration is performed.*

Furthermore the terms which are dependent on the wave packet structure (Y and X) vanish when $m_{\beta,N} = m_{\alpha,N}$ (which is always the case). In this case one obtains finally [6]

$$\begin{aligned}&\Delta\phi(t_D) = -\frac{p_{\alpha1}+p_{\beta1}}{2\hbar}(q_{\beta1}-q_{\alpha1})\\ &\quad + \sum_{i=1}^{N}\left[(k_{\beta i}-k_{\alpha i})\frac{q_{\alpha i}+q_{\beta i}}{2} - (\omega_{\beta i}-\omega_{\alpha i})t_i + \varphi_{\beta i} - \varphi_{\alpha i}\right]\\ &\quad + \sum_{i=1}^{N}\left(\frac{m_{\beta i}-m_{\alpha i}}{2\hbar}\right)\left\{\left(\frac{S_{\alpha i}}{m_{\alpha i}} + \frac{p_{\alpha,i+1}}{2m_{\alpha i}}(q_{\beta,i+1}-q_{\alpha,i+1})\right.\right.\\ &\quad \left. - \frac{p_{\alpha i}+\hbar k_{\alpha i}}{2m_{\alpha i}}(q_{\beta i}-q_{\alpha i})\right) + \left(\frac{S_{\beta i}}{m_{\beta i}}\right.\\ &\quad \left.\left. + \frac{p_{\beta,i+1}}{2m_{\beta i}}(q_{\alpha,i+1}-q_{\beta,i+1}) - \frac{p_{\beta i}+\hbar k_{\beta i}}{2m_{\beta i}}(q_{\alpha i}-q_{\beta i})\right)\right\}. \qquad (14)\end{aligned}$$

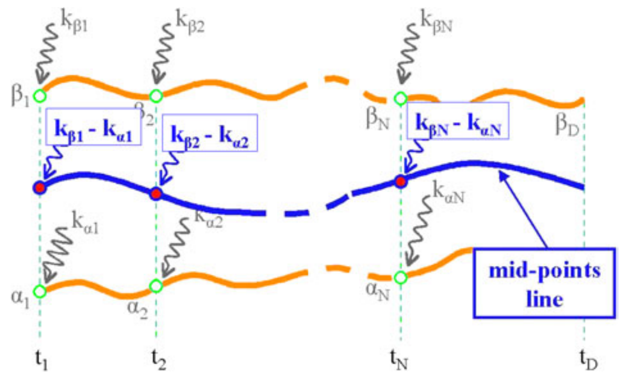

Figure 3. Equivalent mid-point line.

This formula gives the phase shift in the general case of different masses on both arms after integration over the detection region.

5.2. Identical masses

The case of identical masses is an important approximation which is commonly used for the modelization of many devices like gravimeters and gyrometers [8, 9, 12]. If $m_{\alpha i} = m_{\beta i} = m, \forall i$, this general phase shift becomes [6]

$$\Delta\phi(t_D) = -\frac{p_{\alpha 1} + p_{\beta 1}}{2\hbar}(q_{\beta 1} - q_{\alpha 1}) + \sum_{i=1}^{N}(k_{\beta i} - k_{\alpha i})\frac{q_{\alpha i} + q_{\beta i}}{2} + \sum_{i=1}^{N}[\varphi_{\beta i} - \varphi_{\alpha i} - (\omega_{\beta i} - \omega_{\alpha i})t_i] \quad (15)$$

which gives when $q_{\beta 1} = q_{\alpha 1}$ (always the case)

$$\Delta\phi(t_D) = \sum_{i=1}^{N}\left[(k_{\beta i} - k_{\alpha i})\frac{q_{\alpha i} + q_{\beta i}}{2} + (\varphi_{\beta i} - \varphi_{\alpha i}) - (\omega_{\beta i} - \omega_{\alpha i})t_i\right]. \quad (16)$$

This result depends on the mid-points $(q_{\alpha i} + q_{\beta i})/2$ and on the differences of interaction parameters ($k_{\beta i} - k_{\alpha i}$, $\varphi_{\beta i} - \varphi_{\alpha i}$, $\omega_{\beta i} - \omega_{\alpha i}$), and leads to the conclusion that any interferometer can be seen as a line of classical mid-points with effective interactions (where the wavevectors of these interactions are equal to the difference of individual wavevectors) (see figure 3).

5.3. Symmetrical geometry

We can also specify the form of the phase shift (16) when the interferometer geometry is symmetrical (see figure 4).

The considered symmetry is defined as $k_{\beta i} + k_{\alpha i} = 0$, $\forall i \in [2, N-1]$, i.e. it is a symmetry with respect to the direction of the particular vector: $p_{initial} + \hbar k_{initial}/2$.

Consequently,

$$\Delta\phi(t_N^+) = k_1 q_1 + 2\sum_{i=2}^{N-1} k_i \frac{q_{\alpha i} + q_{\beta i}}{2} + k_N \frac{q_{\alpha N} + q_{\beta N}}{2} - \sum_{i=1}^{N}(\varphi_{\beta i} - \varphi_{\alpha i}). \quad (17)$$

But $\forall i \in [1, N-1]$,

$$\frac{q_{\alpha,i+1} + q_{\beta,i+1}}{2} = \xi_{i+1,i} + A_{i+1,i}\frac{q_{\alpha i} + q_{\beta i}}{2} + \frac{B_{i+1,i}}{m}\frac{p_{\alpha i} + p_{\beta i}}{2} \quad (18)$$

$$= \xi_{i+1,1} + A_{i+1,1}q_1 + \frac{B_{i+1,1}}{m}\left(p_1 + \frac{\hbar k_1}{2}\right) \quad (19)$$

$$:= Q(t_{i+1}) \quad (20)$$

which can be calculated with the $ABCD\xi$ law (see [2]).

It depends only on q_1 ('central position' of the first interaction, see section 2.2 and [5]) and $p_1 + \hbar k_1/2$ ('Bragg initial momentum').

Therefore [6],

$$\Delta\phi(t_N^+) = \sum_{i=1}^{N}[(k_{\beta i} - k_{\alpha i})Q(t_i) + \varphi_{\alpha i} - \varphi_{\beta i}] \quad (21)$$

which has a very simple form when the Bragg condition is fulfilled ($p_1 + \hbar k_1/2 = 0$).

6. *ABCD* matrices for a time-dependent Hamiltonian

At this point, we have to introduce the expressions of the *ABCD* matrices and of the vector ξ (see equation (3)) which enter the formulae (13), (14), (16) and (19).

6.1. General case

The most general case that we consider in this paper is defined by the time-dependent Hamiltonian H_{ext} (see (1)).

Hamilton's equations are in this case

$$\dot{\chi} = \Gamma(t)\chi + \Phi(t) \quad (22)$$

with

$$\chi := \begin{pmatrix} q \\ p/m \end{pmatrix} \quad (23)$$

$$\Phi(t) := \begin{pmatrix} 0 \\ g(t) \end{pmatrix} \quad (24)$$

and a time-dependent 6×6 matrix:

$$\Gamma(t) := \begin{pmatrix} \alpha(t) & \beta(t) \\ \gamma(t) & \delta(t) \end{pmatrix} \quad (25)$$

with

$$\delta(t) = \alpha(t) = \mathrm{i}\vec{J}\cdot\vec{\Omega}(t) = \begin{pmatrix} 0 & \Omega_z(t) & -\Omega_y(t) \\ -\Omega_z(t) & 0 & \Omega_x(t) \\ \Omega_y(t) & -\Omega_x(t) & 0 \end{pmatrix}. \quad (26)$$

This differential equation has an exact solution which is

$$\chi_{21} := \chi(t_2, t_1) = \begin{pmatrix} A_{21} & B_{21} \\ C_{21} & D_{21} \end{pmatrix} \times \left[\chi(t_1) + \int_{t_1}^{t_2}\begin{pmatrix} A(t', t_1) & B(t', t_1) \\ C(t', t_1) & D(t', t_1) \end{pmatrix}^{-1}\Phi(t')\,\mathrm{d}t'\right] \quad (27)$$

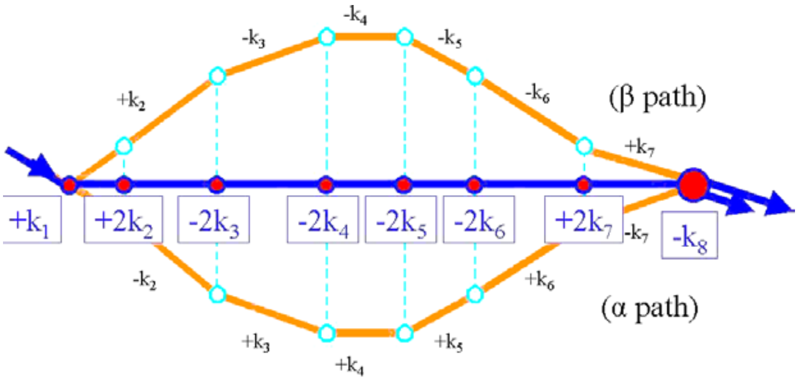

Figure 4. A typical symmetrical interferometer with its equivalent mid-point line.

where ($\mathcal{T}$ is the time-ordering operator)

$$\begin{pmatrix} A_{21} & B_{21} \\ C_{21} & D_{21} \end{pmatrix} = \mathcal{T}\left[\exp\left(\int_{t_1}^{t_2} \Gamma(t')\,\mathrm{d}t'\right)\right]$$
$$:= \prod_{dt_j\to 0} \exp(\Gamma(t_j)\,\mathrm{d}t_j)$$
$$= 1 + \int_{t_1}^{t_2} \Gamma(t')\,\mathrm{d}t' + \int_{t_1}^{t_2}\int_{t_1}^{t'} \Gamma(t')\Gamma(t'')\,\mathrm{d}t'\,\mathrm{d}t'' + \cdots. \quad (28)$$

From (27) and (3), one obtains the expression of ξ:

$$\mathcal{R}_{21}\xi_{21} = (A_{21} \quad B_{21}) \int_{t_1}^{t_2} \begin{pmatrix} A(t', t_1) & B(t', t_1) \\ C(t', t_1) & D(t', t_1) \end{pmatrix}^{-1} \Phi(t')\,\mathrm{d}t'$$
$$= (A_{21} \quad B_{21}) \int_{t_1}^{t_2} \begin{pmatrix} -\tilde{B}(t', t_1)D(t', t_1)B^{-1}(t', t_1)g(t') \\ B^{-1}(t', t_1)A(t', t_1)\tilde{B}(t', t_1)g(t') \end{pmatrix} \mathrm{d}t' \quad (29)$$

with

$$\mathcal{R}_{21} = \mathcal{R}(t_2, t_1) := \mathcal{T}\left[\exp\left(\int_{t_1}^{t_2} \alpha(t')\,\mathrm{d}t'\right)\right]. \quad (30)$$

Finally, we can replace the *ABCD* matrices (and ξ) by these expressions in formulae (13), (14), (16) and (19), to obtain analytical expressions of the phase shift. It is then possible to expand the time-ordered exponentials in powers of α ($\alpha := \mathrm{i}\vec{J}\cdot\vec{\Omega}$), β and γ to any wanted accuracy.

Now let us see the relevant case of time-independent Hamiltonians.

6.2. Time-independent Hamiltonian

When $\vec{\vec{\beta}}$, $\vec{\vec{\gamma}}$, $\vec{\Omega}$ and $\vec{g}$ are independent of time, one obtains

$$\begin{pmatrix} A_{21} & B_{21} \\ C_{21} & D_{21} \end{pmatrix} = \mathrm{e}^{\Gamma(t_2-t_1)} \quad (31)$$

and

$$\mathcal{R}_{21}\xi_{21} = (1 \quad 0)\,\Gamma^{-1}(\mathrm{e}^{\Gamma(t_2-t_1)} - 1)\begin{pmatrix} 0 \\ g \end{pmatrix}$$
$$= (1 - A_{21} + B_{21}\beta^{-1}\alpha)(\alpha\beta^{-1}\alpha - \gamma)^{-1}g. \quad (32)$$

This particular case corresponds, for example, to an atom interferometer laid out on the ground of the Earth. In the ground reference frame, the atoms are submitted to a field of gravity ($\vec{g}$ and $\vec{\vec{\gamma}}$ on the ground) and to a rotation, both of which do not depend on time (in a first approach). The above calculations can then be applied, and the phase shift expression can be expanded in a Taylor series about Γt_1 to any desired order (see appendix A).

7. Application to symmetrical Ramsey–Bordé atom interferometers

Let us apply the previous results to a symmetrical Ramsey–Bordé atom interferometer [1, 11] (Mach–Zehnder geometry). We shall detail two particular cases and establish the link with well-known perturbative results.

7.1. The Mach–Zehnder geometry

This geometry (see figures 5 and 6) corresponds to the case described in section 5.3 and leads to (see equation (21))

$$\Delta\phi(t_3^+) = k_1q_1 - 2k_2Q(t_2) + k_3Q(t_3) + (\varphi_1 - 2\varphi_2 + \varphi_3) \quad (33)$$

that is

$$\Delta\phi(t_3^+) = (k_1 + k_3A_{31} - 2k_2A_{21})q_1 + (k_3B_{31} - 2k_2B_{21})v_1$$
$$+ k_3\mathcal{R}_{31}\xi_{31} - 2k_2\mathcal{R}_{21}\xi_{21} + (\varphi_1 - 2\varphi_2 + \varphi_3) \quad (34)$$

with $v_1 := p_1/m + \hbar k_1/2m$, and where A, B and ξ are given by the expressions (28) and (29).

7.2. Specific case 1: gravity + gradient of gravity + rotation in the frame where the lasers are at rest

Let us see the particular case of such an interferometer when a field of gravity (g) plus a gradient of gravity (γ) plus a rotation (Ω) act in the frame where the lasers are at rest. We consider a time-independent Hamiltonian with $\beta = 1$ (the phase shift formula is valid for a time-dependent Hamiltonian but here, in a first approach, only the time-independent effects are considered), and we take $\vec{k}_3 = \vec{k}_2 = \vec{k}_1$.

With $t_1 = 0$ (for simplicity), $T = t_2 - t_1$ and $T' = t_3 - t_2$, we obtain

$$\Delta\phi(T + T') = k_1(1 + A(T + T') - 2A(T))q_1$$
$$+ k_1(B(T + T') - 2B(T))v_1$$
$$+ k_1(\mathcal{R}(T + T')\xi(T + T') - 2\mathcal{R}(T)\xi(T))$$
$$+ (\varphi_1 - 2\varphi_2 + \varphi_3). \quad (35)$$

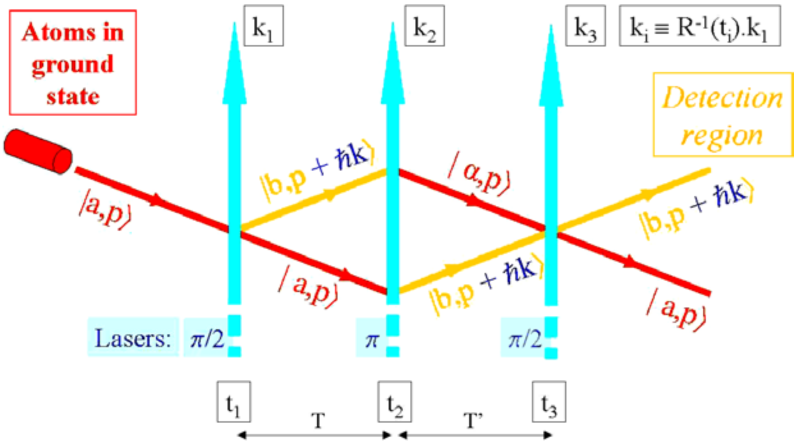

Figure 5. A symmetrical Ramsey–Bordé interferometer.

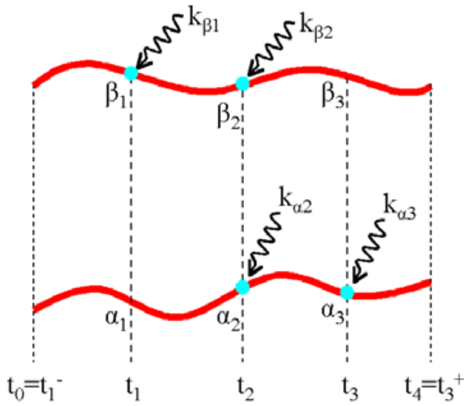

Figure 6. Space–time geometry of a symmetrical Ramsey–Bordé interferometer.

If we replace ξ by its expression (32) established in section 6.2, this result becomes

$$\begin{aligned}\Delta\phi(T+T') &= k_1(1+A(T+T')-2A(T))[q_1-(\alpha^2-\gamma)^{-1}g] \\ &\quad + k_1(B(T+T')-2B(T))[v_1+\alpha(\alpha^2-\gamma)^{-1}g] \\ &\quad + (\varphi_1-2\varphi_2+\varphi_3)\end{aligned} \tag{36}$$

where we can also replace the matrices A and B by their expression (31).

This result takes into account the effect of gravity + a gradient of gravity + a rotation (time-independent effects in this specific case) in an exact way (gyro-accelerometer). It depends on the coordinates of the rotation centre through q_1 (q_1 is a vector which links the central position of the first interaction (see section 2.2) to the rotation centre where the origin of coordinates has been taken), and on the two basic vector parameters $q_1-(\alpha^2-\gamma)^{-1}g$ and $p_1+\hbar k_1/2+m\alpha(\alpha^2-\gamma)^{-1}g$.

We can also make a Taylor expansion of this exact phase shift in powers of Ω (rotation rate) and γ in order to discuss the importance of each phase shift contribution (see appendix A).

7.3. *Specific case 2: gravity + gradient of gravity + rotation of the lasers*

In this part, we consider the particular case of an atom interferometer which is subjected to a time-independent field of gravity (g and gradient γ) in the laboratory frame and where a rotation acts on the beam splitters (an artificial rotation for example). Consequently, in the laboratory frame (see [2]),

$$A(T) = \cosh(\gamma^{1/2}T) \tag{37}$$

$$B(T) = \gamma^{-1/2}\sinh(\gamma^{1/2}T) \tag{38}$$

$$\xi(T) = \gamma^{-1}(\cosh(\gamma^{1/2}T)-1)g \tag{39}$$

and, on the other hand, we can use the rotation matrix $\mathcal{R}(t_i)$ which links the fixed laser wavevectors k_{0i} to the rotated k_i:

$$k_{0i} = \mathcal{R}(t_i)k_i. \tag{40}$$

In an ideal interferometer we may take $k_{0i} = k_1, \forall i$. We get

$$k_1 = \mathcal{R}(T)k_2 = \mathcal{R}(T+T')k_3 \tag{41}$$

which leads to

$$\begin{aligned}\Delta\phi(T+T') &= -k_1[1-2\mathcal{R}(T)+\mathcal{R}(T+T')]\gamma^{-1}g \\ &\quad + k_1(1-2\mathcal{R}(T)A(T)+\mathcal{R}(T+T')A(T+T')) \\ &\quad \times [q_1+\gamma^{-1}g] + k_1(-2\mathcal{R}(T)B(T) \\ &\quad + \mathcal{R}(T+T')B(T+T'))v_1 + \varphi_1 - 2\varphi_2 + \varphi_3\end{aligned} \tag{42}$$

which depends on the two vector parameters $q_1+\gamma^{-1}g$ and $p_1+\frac{\hbar k_1}{2}$.

We can also replace A and B by their previous analytic expression and expand them in powers of Ω and γ to obtain finally the Taylor expansion (in ΩT and γT^2) of the atomic phase shift (see appendix B).

7.4. *Link to well-known results*

For simplicity, we no longer write the term $\varphi_1-2\varphi_2+\varphi_3$.

7.4.1. Gyrometer. If we neglect the effect of gravity (and its gradient) the following result is obtained ($T'=T$):

$$\Delta\phi(2T) = k_1q_1 - 2k_1\mathcal{R}(T)[q_1+v_1T] + k_1\mathcal{R}(2T)[q_1+2v_1T] \tag{43}$$

where $v_1 := p_1/m + \hbar k_1/2m$.

Let us express $\mathcal{R}(T)$ (with $\vec{\Omega} = \Omega\vec{n}$):

$$\begin{aligned}\mathcal{R}(\vec{n},\Omega T) &:= e^{i\Omega T\vec{n}\cdot\vec{J}} \\ &= 1 + i\vec{n}\cdot\vec{J}\sin(\Omega T) - (\vec{n}\cdot\vec{J})^2(1-\cos(\Omega T))\end{aligned} \tag{44}$$

which leads to

$$\mathcal{R}^{-1}(\vec{n},\Omega T)\cdot\vec{k} = \vec{k}+\sin(\Omega T)\vec{n}\times\vec{k}+(\cos(\Omega T)-1)\vec{k}_\perp \quad (45)$$

where $\vec{k}_\perp := \vec{k} - (\vec{n}\cdot\vec{k})\vec{n}$.

Finally one obtains

$$\begin{aligned}\Delta\phi(2T) &= \cos(2\Omega T)\vec{k}_\perp\cdot\vec{v}_1 T + \sin(2\Omega T)\vec{n}\cdot(\vec{k}_\perp\times\vec{v}_1 T)\\ &+ \vec{k}_\perp\cdot\vec{q}_1 + [\cos(2\Omega T) - 2\cos(\Omega T)]\vec{k}_\perp\cdot(\vec{q}_1+\vec{v}_1 T)\\ &+ [\sin(2\Omega T) - 2\sin(\Omega T)]\vec{n}\cdot[\vec{k}_\perp\times(\vec{q}_1+\vec{v}_1 T)] \qquad (46)\end{aligned}$$

where one recognizes the result of [3].

In Cartesian coordinates, with $\vec{\Omega} = \Omega\cdot\vec{e}_z$, $\vec{k}_1 = k_1\cdot\vec{e}_y$ and $\vec{p}_1 = p_1\cdot\vec{e}_x$, one obtains to the first order in Ω (taking $q_1 = 0$ for simplicity)

$$\Delta\phi(2T) = 2\Omega T^2 k_1\frac{p_1}{m} + \mathrm{O}((\Omega T)^2) \qquad (47)$$

which gives the well known first-order Sagnac effect.

7.4.2. Gradio-gravimeter. In the particular case of '$g+\gamma$', one arrives at

$$\begin{aligned}&\Delta\phi(T+T')\\ &= k_1[1+\cosh(\gamma^{\frac{1}{2}}(T+T')) - 2\cosh(\gamma^{\frac{1}{2}}T)](q_1+\gamma^{-1}g)\\ &+ k_1\gamma^{-\frac{1}{2}}[\sinh(\gamma^{\frac{1}{2}}(T+T')) - 2\sinh(\gamma^{\frac{1}{2}}T)]\\ &\times\left(\frac{p_1}{m}+\frac{\hbar k_1}{2m}\right) \qquad (48)\end{aligned}$$

which is the 3D generalization of the 1D formula given in [2].

To first order in γ and for $T' = T$, this formula gives (see [2, 8, 12, 13])

$$\begin{aligned}\Delta\phi(2T) &= k_1 g T^2 + k_1\gamma T^2\\ &\times\left(q_1 + \frac{7}{12}T^2 g + \frac{1}{m}\left(p_1+\frac{\hbar k_1}{2}\right)T\right) + \mathrm{O}((\gamma T^2)^2). \quad (49)\end{aligned}$$

8. Application to atomic clocks

The *ABCDξ* formalism provides a unified framework for gravito-inertial sensors as well as for atomic clocks (see [3]). As any microwave or optical atomic clock can be seen as an interferometer (fountain geometry, asymetrical Ramsey–Bordé interferometers, ...), we can apply in this case all the previous results and more particularly the general expression of atomic phase shifts given in section 4 (equation (13)). Thanks to this formula we were able to retrieve the results of [3].

9. Conclusion

This paper draws on the two theorems given in [6] to express the phase shift of atom interferometers which have an arbitrary spatial or temporal beam-splitter configuration. These theorems, established in the framework of the *ABCDξ* formulation of atom optics and of the ttt theorem, are valid for a time-dependent Hamiltonian at most quadratic in position and momentum operators. The first theorem gives a compact expression of the action difference between two homologous paths, and the second one gives an analytical expression of the global phase shift of atom interferometers in the case of such a Hamiltonian.

Any interferometer geometry can be treated with these expressions (gravimeters, gyrometers, microwave and optical atomic clocks, ...). In this paper we have detailed only the particular case of temporal symmetrical Ramsey–Bordé atom interferometers (gyro-accelerometers).

As these analytical expressions give the exact phase shift due to a Hamiltonian at most quadratic, we can calculate perturbatively the effect of a higher order term (necessary for space missions like HYPER [7]). For example it becomes possible to calculate exactly the global phase shift due to gravity plus a gradient of gravity plus a rotation, and then calculate perturbatively the effect of a gradient of gradient of gravity.

Appendix A

In this appendix, we develop expressions applicable to the specific case 1: 'gravity + gradient of gravity + rotation in the frame where the lasers are at rest' (see section 7.2). We perform a Taylor expansion of the phase shift expression (36) in powers of Ω (rotation rate) and γ (gradient of gravity). We take $T = T'$, $t_1 = 0$ and no longer write the term $\varphi_1 - 2\varphi_2 + \varphi_3$.

This phase shift can be rewritten as

$$\Delta\phi(2T) = \Delta\phi_q + \Delta\phi_v + \Delta\phi_g \qquad (50)$$

with

$$\Delta\phi_q := \vec{k}_1\cdot(1 + A(2T) - 2A(T))\cdot\vec{q}_1 \qquad (51)$$

for the part which involves the central position $\vec{q}_1$ of the first beam splitter (see section 2.2 and the remark at the end of section 7.2),

$$\Delta\phi_v := \vec{k}_1\cdot(B(2T) - 2B(T))\cdot\vec{v}_1 \qquad (52)$$

for the part which involves the momentum $m\vec{v}_1 := \vec{p}_1 + \hbar\vec{k}_1/2$ and

$$\begin{aligned}\Delta\phi_g &:= \vec{k}_1\cdot[(B(2T) - 2B(T))\,\alpha\\ &-(1 + A(2T) - 2A(T))](\alpha^2-\gamma)^{-1}\cdot\vec{g} \qquad (53)\end{aligned}$$

for the part which involves the gravity field $\vec{g}$.

We can then replace the matrices A and B by their expression (31) and expand them in powers of α and γ:

$$\begin{aligned}A(T) &= 1 + T\alpha + \frac{T^2}{2!}(\alpha^2+\gamma) + \frac{T^3}{3!}(\alpha^3 + 2\alpha\gamma + \gamma\alpha)\\ &+ \frac{T^4}{4!}(\alpha^4 + 3\alpha^2\gamma + 2\alpha\gamma\alpha + \gamma\alpha^2 + \gamma^2)\\ &+ \frac{T^5}{5!}(\alpha^5 + 4\alpha^3\gamma + 3\alpha^2\gamma\alpha + 2\alpha\gamma\alpha^2 + \gamma\alpha^3\\ &+ 2\alpha\gamma^2 + 2\gamma\alpha\gamma + 2\gamma^2\alpha) + \cdots \qquad (54)\end{aligned}$$

$$\begin{aligned}B(T) &= T + \frac{T^2}{2!}(2\alpha) + \frac{T^3}{3!}(3\alpha^2+\gamma)\\ &+ \frac{T^4}{4!}(4\alpha^3 + 2\alpha\gamma + 2\gamma\alpha)\\ &+ \frac{T^5}{5!}(5\alpha^4 + 3\alpha^2\gamma + 4\alpha\gamma\alpha + 3\gamma\alpha^2 + \gamma^2)\\ &+ \frac{T^6}{6!}(6\alpha^5 + 4\alpha^3\gamma + 6\alpha^2\gamma\alpha + 6\alpha\gamma\alpha^2 + 4\gamma\alpha^3\\ &+ 2\alpha\gamma^2 + 2\gamma\alpha\gamma + 2\gamma^2\alpha) + \cdots. \qquad (55)\end{aligned}$$

For each part of the previous phase shift we obtain consequently (let us recall that α is defined such that $\alpha \cdot \vec{q} := -\vec{\Omega} \times \vec{q}$ for any vector $\vec{q}$)

$$\Delta\phi_q = \vec{k}_1 \cdot [T^2(\alpha^2+\gamma) + T^3(\alpha^3 + 2\alpha\gamma + \gamma\alpha) + \tfrac{7}{12}T^4(\alpha^4 + 3\alpha^2\gamma + 2\alpha\gamma\alpha + \gamma\alpha^2 + \gamma^2) + \tfrac{1}{4}T^5(\alpha^5 + 4\alpha^3\gamma + 3\alpha^2\gamma\alpha + 2\alpha\gamma\alpha^2 + \gamma\alpha^3 + 2\alpha\gamma^2 + 2\gamma\alpha\gamma + 2\gamma^2\alpha) + \cdots] \cdot \vec{q}_1 \quad (56)$$

$$\Delta\phi_v = \vec{k}_1 \cdot [2T^2\alpha + T^3(3\alpha^2+\gamma) + \tfrac{7}{12}T^4(4\alpha^3 + 2\alpha\gamma + 2\gamma\alpha) + \tfrac{1}{4}T^5(5\alpha^4 + 3\alpha^2\gamma + 4\alpha\gamma\alpha + 3\gamma\alpha^2 + \gamma^2) + \tfrac{31}{360}T^6(6\alpha^5 + 4\alpha^3\gamma + 6\alpha^2\gamma\alpha + 6\alpha\gamma\alpha^2 + 4\gamma\alpha^3 + 2\alpha\gamma^2 + 2\gamma\alpha\gamma + 2\gamma^2\alpha) + \cdots] \cdot \vec{v}_1 \quad (57)$$

$$\Delta\phi_g = \vec{k}_1 \cdot [T^2 + 2T^3\alpha + \tfrac{7}{12}T^4(3\alpha^2+\gamma) + \tfrac{1}{4}T^5(4\alpha^3 + 2\alpha\gamma + 2\gamma\alpha) + \tfrac{31}{360}T^6(5\alpha^4 + 3\alpha^2\gamma + 4\alpha\gamma\alpha + 3\gamma\alpha^2 + \gamma^2) + \cdots] \cdot \vec{g}. \quad (58)$$

Let us recall that the initial atomic momentum (divided by m) $\vec{p}_1/m$ is different from the initial atomic velocity taken in the non-rotating frame $\vec{v} + \vec{\Omega} \times \vec{q}_1$.

Finally let us emphasize that, since the phase shift expression (36) is exact, we can get a result in powers of α and γ to any desired accuracy. Practically, we recover all main effects to the lowest orders: gravity (involving $\vec{g}$ and $\vec{\vec{\gamma}}$ only) and Sagnac terms (involving $\vec{\Omega}$ only). In addition, we have a number of crossed terms which should be taken into account in present accurate experiments.

Appendix B

This second appendix focuses on the specific case 2: 'gravity + gradient of gravity + rotation of the lasers' described in section 7.3.

As in appendix A, we make a Taylor expansion of the phase shift expression (42) in powers of Ω (rotation rate for the lasers) and γ (gradient of gravity). We also take $T = T'$ and $t_1 = 0$.

Equation (42) is written as

$$\Delta\phi(2T) := \Delta\phi_q + \Delta\phi_v + \Delta\phi_g \quad (59)$$

where

$$\Delta\phi_q := \vec{q}_1 \cdot [1 + A(2T)\cdot\tilde{\mathcal{R}}(2T) - 2A(T)\cdot\tilde{\mathcal{R}}(T)]\cdot\vec{k}_1 \quad (60)$$

$$\Delta\phi_v := \vec{v}_1 \cdot [B(2T)\cdot\tilde{\mathcal{R}}(2T) - 2B(T)\cdot\tilde{\mathcal{R}}(T)]\cdot\vec{k}_1 \quad (61)$$

$$\Delta\phi_g := \vec{g}\cdot\vec{\vec{\gamma}}^{-1}[(A(2T)-1)\cdot\tilde{\mathcal{R}}(2T) - 2(A(T)-1)\cdot\tilde{\mathcal{R}}(T)]\cdot\vec{k}_1 \quad (62)$$

with $\vec{v}_1 := \vec{p}_1/m + \hbar\vec{k}_1/2m$ and (by definition of cosh and sinh)

$$A(T) = \cosh(\vec{\vec{\gamma}}^{\frac{1}{2}}T) = \sum_{i=0}^{\infty} \vec{\vec{\gamma}}^{i}\, \frac{T^{2i}}{(2i)!} \quad (63)$$

$$B(T) = \vec{\vec{\gamma}}^{-\frac{1}{2}} \cdot \sinh(\vec{\vec{\gamma}}^{\frac{1}{2}}T) = \sum_{i=0}^{\infty} \vec{\vec{\gamma}}^{i}\, \frac{T^{2i+1}}{(2i+1)!}. \quad (64)$$

Moreover,

$$\tilde{\mathcal{R}}(\hat{n}, \Omega T)\cdot\vec{k}_1 = \vec{k}_1 + (\cos(\Omega T)-1)\vec{k}_{1\perp} + \sin(\Omega T)\hat{n}\times\vec{k}_1 \quad (65)$$

with $\vec{k}_{1\perp} := \vec{k}_1 - (\vec{k}_1\cdot\hat{n})\hat{n}$, and where $\Omega := \sqrt{\Omega_x^2 + \Omega_y^2 + \Omega_z^2}$ is the rotation rate and $\hat{n} := \begin{pmatrix}\Omega_x/\Omega\\ \Omega_y/\Omega\\ \Omega_z/\Omega\end{pmatrix}$ the unitary rotation vector.

A Taylor expansion in ΩT gives

$$\tilde{\mathcal{R}}(\hat{n}, \Omega T)\cdot\vec{k}_1 = \vec{k}_1 + \sum_{j=1}^{\infty}(-1)^j \frac{(\Omega T)^{2j}}{(2j)!}\vec{k}_{1\perp} + \sum_{j=0}^{\infty}(-1)^j \frac{(\Omega T)^{2j+1}}{(2j+1)!}(\hat{n}\times\vec{k}_1). \quad (66)$$

Consequently one obtains

$$\Delta\phi_q = \sum_{i=1}^{\infty}[\vec{q}_1\cdot\vec{\vec{\gamma}}^{i}\cdot\vec{k}_1]T^{2i}\left(\frac{2^{2i}-2}{(2i)!}\right) + \sum_{i=0}^{\infty}\sum_{j=1}^{\infty}[\vec{q}_1\cdot\vec{\vec{\gamma}}^{i}\cdot\vec{k}_{1\perp}]\Omega^{2j}T^{2i+2j}\left((-1)^j\frac{2^{2i+2j}-2}{(2i)!(2j)!}\right) + \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}[\vec{q}_1\cdot\vec{\vec{\gamma}}^{i}\cdot(\hat{n}\times\vec{k}_1)]\Omega^{2j+1}T^{2i+2j+1} \times \left((-1)^j\frac{2^{2i+2j+1}-2}{(2i)!(2j+1)!}\right) \quad (67)$$

for the piece that involves the position of the first beam splitter $\vec{q}_1$,

$$\Delta\phi_v = \sum_{i=0}^{\infty}[\vec{v}_1\cdot\vec{\vec{\gamma}}^{i}\cdot\vec{k}_1]T^{2i+1}\left(\frac{2^{2i+1}-2}{(2i+1)!}\right) + \sum_{i=0}^{\infty}\sum_{j=1}^{\infty}[\vec{v}_1\cdot\vec{\vec{\gamma}}^{i}\cdot\vec{k}_{1\perp}]\Omega^{2j}T^{2i+2j+1}\left((-1)^j\frac{2^{2i+2j+1}-2}{(2i+1)!(2j)!}\right) + \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}[\vec{v}_1\cdot\vec{\vec{\gamma}}^{i}\cdot(\hat{n}\times\vec{k}_1)]\Omega^{2j+1}T^{2i+2j+2} \times \left((-1)^j\frac{2^{2i+2j+2}-2}{(2i+1)!(2j+1)!}\right) \quad (68)$$

for the piece that involves the initial momentum $\vec{p}_1$ and

$$\Delta\phi_g = \sum_{i=0}^{\infty}[\vec{g}\cdot\vec{\vec{\gamma}}^{i}\cdot\vec{k}_1]T^{2i+2}\left(\frac{2^{2i+2}-2}{(2i+2)!}\right) + \sum_{i=0}^{\infty}\sum_{j=1}^{\infty}[\vec{g}\cdot\vec{\vec{\gamma}}^{i}\cdot\vec{k}_{1\perp}]\Omega^{2j}T^{2i+2j+2}\left((-1)^j\frac{2^{2i+2j+2}-2}{(2i+2)!(2j)!}\right) + \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}[\vec{g}\cdot\vec{\vec{\gamma}}^{i}\cdot(\hat{n}\times\vec{k}_1)]\Omega^{2j+1}T^{2i+2j+3} \times \left((-1)^j\frac{2^{2i+2j+3}-2}{(2i+2)!(2j+1)!}\right) \quad (69)$$

for the piece that involves the gravity field $\vec{g}$.

The first terms of these Taylor expansions are

$$\Delta\phi_q = -(\vec{q}_1\cdot\vec{k}_{1\perp})\Omega^2T^2 + (\vec{q}_1\cdot\vec{\vec{\gamma}}\cdot\vec{k}_1)T^2 - [\vec{q}_1\cdot(\hat{n}\times\vec{k}_1)]\Omega^3T^3 + 3[\vec{q}_1\cdot\vec{\vec{\gamma}}\cdot(\hat{n}\times\vec{k}_1)]\Omega T^3 + \tfrac{7}{12}(\vec{q}_1\cdot\vec{k}_{1\perp})\Omega^4T^4 + \tfrac{7}{12}(\vec{q}_1\cdot\vec{\vec{\gamma}}^2\cdot\vec{k}_1)T^4 + \tfrac{1}{4}[\vec{q}_1\cdot(\hat{n}\times\vec{k}_1)]\Omega^5T^5 - \tfrac{5}{2}[\vec{q}_1\cdot\vec{\vec{\gamma}}\cdot(\hat{n}\times\vec{k}_1)]\Omega^3T^5 - \tfrac{31}{24}(\vec{q}_1\cdot\vec{\vec{\gamma}}^2\cdot\vec{k}_{1\perp})\Omega^2T^6 + \cdots \quad (70)$$

$$\begin{aligned}\Delta\phi_v &= 2[\vec{v}_1\cdot(\hat{n}\times\vec{k}_1)]\Omega T^2 - 3(\vec{v}_1\cdot\vec{k}_{1\perp})\Omega^2 T^3 \\ &+ (\vec{v}_1\cdot\overset{\Rightarrow}{\gamma}\cdot\vec{k}_1)T^3 - \tfrac{7}{3}[\vec{v}_1\cdot(\hat{n}\times\vec{k}_1)]\Omega^3 T^4 \\ &+ \tfrac{7}{3}[\vec{v}_1\cdot\overset{\Rightarrow}{\gamma}\cdot(\hat{n}\times\vec{k}_1)]\Omega T^4 + \tfrac{5}{4}(\vec{v}_1\cdot\vec{k}_{1\perp})\Omega^4 T^5 \\ &- \tfrac{5}{2}(\vec{v}_1\cdot\overset{\Rightarrow}{\gamma}\cdot\vec{k}_{1\perp})\Omega^2 T^5 + \tfrac{1}{4}(\vec{v}_1\cdot\overset{\Rightarrow}{\gamma}^2\cdot\vec{k}_1)T^5 \\ &- \tfrac{31}{18}[\vec{v}_1\cdot\overset{\Rightarrow}{\gamma}\cdot(\hat{n}\times\vec{k}_1)]\Omega^3 T^6 + \cdots \end{aligned} \quad (71)$$

$$\begin{aligned}\Delta\phi_g &= (\vec{g}\cdot\vec{k}_1)T^2 - 3[\vec{g}\cdot(\hat{n}\times\vec{k}_1)]\Omega T^3 - \tfrac{7}{2}(\vec{g}\cdot\vec{k}_{1\perp})\Omega^2 T^4 \\ &+ \tfrac{7}{12}(\vec{g}\cdot\overset{\Rightarrow}{\gamma}\cdot\vec{k}_1)T^4 + \tfrac{5}{2}[\vec{g}\cdot(\hat{n}\times\vec{k}_1)]\Omega^3 T^5 \\ &- \tfrac{5}{4}[\vec{g}\cdot\overset{\Rightarrow}{\gamma}\cdot(\hat{n}\times\vec{k}_1)]\Omega T^5 + \tfrac{31}{24}(\vec{g}\cdot\vec{k}_{1\perp})\Omega^4 T^6 \\ &- \tfrac{31}{24}(\vec{g}\cdot\overset{\Rightarrow}{\gamma}\cdot\vec{k}_{1\perp})\Omega^2 T^6 + \tfrac{31}{360}(\vec{g}\cdot\overset{\Rightarrow}{\gamma}^2\cdot\vec{k}_1)T^6 + \cdots \end{aligned} \quad (72)$$

for the parts involving $\vec{q}_1$, $\vec{p}_1 + \hbar\vec{k}_1/2$ and $\vec{g}$ respectively.

As in appendix A, we have pure rotational and gravitational effects as well as many crossed terms.

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