Atom interferometry using internal excitation: Foundations and recent theory

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Summary. — We first give a short historical introduction to the fundamentals of atom interferometry with internal excitation, based on density matrix diagrams and a Liouville space approach. Then, a new framework is proposed to compare and unify photon and atom optics, which rests on the quantization of proper time. A common wave equation written in five dimensions reduces both cases to 5D optics of massless particles. The ordinary methods of optics (eikonal equation, Kirchhoff integral, Lagrange invariant, Fermat principle, symplectic algebra and ABCD matrices...) are used to solve this equation in practical cases. The various phase shift cancellations, which occur in atom interferometers, are then easily explained. A general phase-shift formula for interferometers is derived in five dimensions, which applies to clocks as well as to gravito-inertial sensors. This contribution is an update of a previous presentation of 5D matter-wave optics and interferometry. Electromagnetic interactions are explicitly added in the 5D metric tensor in complete analogy with Kaluza's work. The 5D Lagrangian is rederived and an expression for the Hamiltonian suitable for the parabolic approximation is presented. The corresponding equations of motion are also given. The 5D action is shown to cancel for the actual trajectory which is a null geodesics of the 5D metric. This presentation is mainly devoted to the classical aspects of the theory and only general consequences for the quantum phase of matter-waves are outlined. The application to Bordé-Ramsey interferometers is given as an illustration.

1. – Introduction

Atom interferometry with labelled internal energy states [1] has emerged from high-resolution laser spectroscopy, essentially from a search for the proper combination of sub-Doppler and Ramsey separate fields techniques [4, 5] and as a consequence of recoil physics [6-8]. For a long time the external motion of atoms had been treated classically in spectroscopy. But the exchange of momentum which occurs naturally during the emission or absorption of photons by atoms and molecules leads to a coupling between the internal and external worlds of these objects and quantum mechanics manifests itself in both aspects in a correlated (intricated) way. It became necessary to introduce atomic wave packets with values of their central momentum differing by integer multiples of the photon momentum $\hbar\vec{k}$ [52, 27]. In the beginning the use of standing waves resulted in complex multiple exchanges of momentum with the counterpropagating fields. The use of travelling waves allowed to isolate well-defined interferometers [52, 45, 9, 1, 16, 48]. Any quantum-mechanical process results from the interference of two paths and may be represented by a double Feynman (density matrix) diagram [27]. As a simple example let us consider the diagram corresponding to linear absorption of light by a gas of two-level atoms [36] (fig. 1).

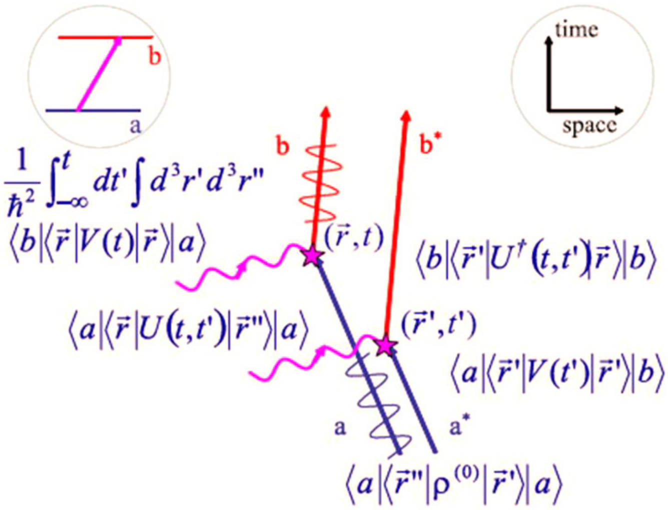

Fig. 1. – A density matrix diagram [27] gives an interferometric representation of the linear absorption of photons by a two-level atom [36]. The ground-state poputation is turned into an off-diagonal density matrix element (optical coherence) by a first interaction with light. This optical coherence is transformed back into a population at a second space-time point. If the delay between the two vertices is fixed, the phase factor associated with the detuning is also fixed and gives rise to Ramsey fringes. In the optical domain, the Doppler phase needs also to be controlled and this is achieved by closing the interferometer with two additional interactions (*e.g.*, as in fig 5).

This process constitutes the archetype of an atom interferometer in which two propagation modes differ by both internal energy (mass) and external motion (momentum) and are coupled by a coherent mode coupler. Ramsey fringes are obtained when the momentum on one arm is modified because of an additional kinetic energy provided by a detuning from the resonance condition [2]. The resulting phase is well-defined only if the distance between the two interaction vertices is fixed. Furthermore, in the optical domain, the splitting between the outgoing wave packets is too large and velocity dependent which blurs out the fringes. This is why the separation of the field zones needs to be associated with a sub-Doppler technique for which the interferometer will close upon itself. Several interferometer architectures satisfy this condition as discussed in references [9, 10] and specifically the Bordé-Ramsey scheme and the photon-echo geometry having, respectivey, two pairs of parallel beam splitters either counterpropagating or copropagating. A first demonstration of the first type was given both with SF_6 molecules [45, 9] and Ca atoms [46]. The sensitivity to inertial and gravitational fields has then been discussed in several places [50, 1, 10, 2] and demonstrated in two early experiments [11, 49]. An overview of these early developments can be found in [3] and especially in [19, 15, 55]. Many new interaction geometries have followed these first architectures in order to increase interferometer areas thanks to multiple interactions and atom cloud levitation [12, 13, 35, 40, 30, 32, 37, 38] or to acquire sensitivity to gravito-inertial fields along several dimensions [25, 29] with numerous terrestrial and spatial applications [31, 51].

A specific tool adapted to density matrix diagrams and hence also to interferometers, is the relativistic Liouville equation in the interaction representation, in which uncoupled free modes are described by free propagators (obtained from wave equations for free fields) and mode couplings by an interaction Liouvillian $\widetilde{\mathcal{L}}_{int}(x)$ and the corresponding propagator for the Liouville vector [27]:

$$|\widetilde{\rho}(\sigma)\rangle\rangle = T \exp\left[-i\int_{\sigma_0}^{\sigma} \mathrm{d}^4x' \widetilde{\mathcal{L}}_{int}(x')\right] |\widetilde{\rho}(\sigma_0)\rangle\rangle,$$

where T is a time-ordering operator, σ and σ_0 space-like hypersurfaces.

In the absence of decoherence in the interferometer $|\widetilde{\rho}(\sigma)\rangle\rangle = |\widetilde{\psi}(\sigma)\widetilde{\psi}^{\dagger}(\sigma)\rangle\rangle$, where the ket and bra vectors correspond to each arm of the interferometer and the previous formula gives directly a relativistic expression of the phase shift of the interferometer. Traditional approaches to atom interferometers use sequences of such evolution operators (free propagators and S-matrices) [48, 21, 19, 34]. These approaches treat the internal states of the atom as an additional degree of freedom superimposed on the external motion without really integrating both aspects in a unified framework. When complex objects such as atoms or molecules interfere with themselves, one should consider that each particle inside the composite object interferes with itself as well. The optical paths of these various individual waves have contributions from the internal motion as well as from the external motion of the entire composite object. In the usual picture, different contributions of the action coming from different discrete masses corresponding to the internal excitations enter an additional phase factor in an artificial way. For a long time the magic cancellation which occurs between the phase coming from the action and the phase originating from the separation of the end points of an interferometer remained mysterious. The new viewpoint, that we suggest now, considers the internal atomic or molecular motion as an additional spatial dimension through which the waves may propagate.

A wave equation written in five dimensions [20, 47, 33] reduces atom optics to 5D optics of massless particles. The ordinary methods of optics (Lagrange invariant, Fermat principle, eikonal equation [22], symplectic algebra and ABCD matrices [14]...) are used to solve this equation in practical cases. The foundations of relativistic 5D optics for matter waves have been presented in an earlier publication [47,33]. This is a natural framework to unify and compare photon and atom optics thanks to formulas valid for arbitrary mass. The concept of mass and its relationship with proper time in terms of associated dynamical variables and conjugate quantum observables are presented again here. Gravito-inertial fields and electromagnetic fields are included in the 5D metric tensor as in Kaluza's theory. A corrected expression is given here for the 5D Lagrangian and corresponding equations of motion are derived. As in 4D, a superaction makes the link with the quantum-mechanical phase in 5D. The 5D generalization of the ABCD theorem [14,21,2] for matter-wave packets leads to a single formula for the quantum phase in the presence of external fields taking into account the internal degrees of freedom of the particle. The various phase shifts, which occur in interferometers, including the effect of gravitational waves [25, 17, 18], are then easily derived and discussed from formulas valid for both relativistic atom waves and optical beams.

2. – The status of mass in classical relativistic mechanics: From 4 to 5 dimensions

In special relativity, the total energy E and the momentum components p^1, p^2, p^3 of a particle, transform as the contravariant components of a four-vector

$$p^\mu = (p^0, p^1, p^2, p^3) = (E/c, \overrightarrow{p}) \tag{1}$$

and the covariant components are given by

$$p_\mu = g_{\mu\nu} p^\mu, \tag{2}$$

where $g_{\mu\nu}$ is the metric tensor. In Minkowski space of signature $(+, -, -, -)$

$$p_\mu = (p_0, p_1, p_2, p_3) = (E/c, -p^1, -p^2, -p^3). \tag{3}$$

These components are conserved quantities when the system considered is invariant under corresponding space-time translations. They will become the generators of space-time translations in the quantum theory. For massive particles of rest mass m, they are connected by the following energy-momentum relation (see fig. 2):

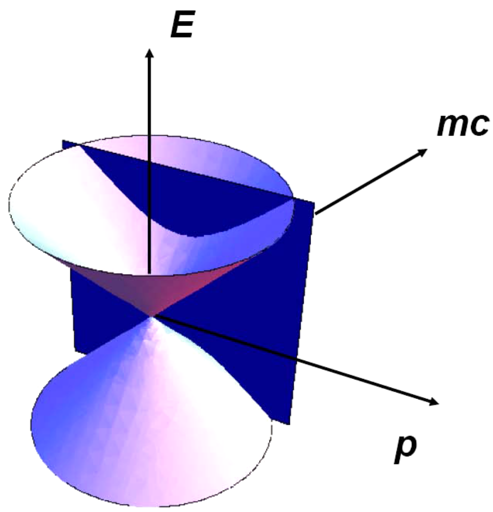

Fig. 2. – 5D energy-momentum picture.

$$E^2 = p^2c^2 + m^2c^4 \tag{4}$$

or, in manifestly covariant form,

$$p^\mu p_\mu - m^2c^2 = 0. \tag{5}$$

This equation cannot be considered as a definition of mass since the origin of mass is not in the external motion but rather in an internal motion (see the Appendix of ref. [44]). It simply relates two relativistic invariants and gives a relativistic expression for the total energy. Thus mass appears as an additional momentum component mc corresponding to internal degrees of freedom of the object and which adds up quadratically with external components of the momentum to yield the total energy squared (Pythagoras' theorem). In the reference frame in which $p = 0$ the mass squared is responsible for the total energy and can thus be seen as stored internal energy just like kinetic energy is a form of external energy. Even when this internal energy is purely kinetic, *e.g.*, in the case of a photon in a box, it appears as pure mass m^* for the global system (*i.e.* the box). This new mass is the relativistic mass of the stored particle

$$m^*c^2 = \sqrt{p^2c^2 + m^2c^4}. \tag{6}$$

The concept of relativistic mass has been criticized in the past but, as we shall see, it becomes relevant for embedded systems. We may have a hierarchy of composed objects (*e.g.* nuclei, atoms, molecules, atomic clocks ...) and at each level the mass m^* of the larger object is given by the sum of energies p^0 of the inner particles. It transforms as p^0 with the internal coordinates and is a scalar with respect to the upper level coordinates.

Mass is conserved when the system under consideration is invariant in a proper time translation and will become the generator of such translations in the quantum theory. In the case of atoms, the internal degrees of freedom give rise to a mass which varies with the internal excitation. For example, in the presence of an electromagnetic field inducing transitions between internal energy levels, the mass of atoms becomes time-dependent (Rabi oscillations). It is thus necessary to enlarge the usual framework of dynamics to introduce this new dynamical variable as a fifth component of the energy-momentum vector.

Equation (5) can be written with a five-dimensional notation

$$G^{\hat{\mu}\hat{\nu}}\widehat{p}_{\hat{\mu}}\widehat{p}_{\hat{\nu}} = 0 \quad \text{with} \quad \hat{\mu}, \hat{\nu} = 0, 1, 2, 3, 4, \tag{7}$$

where $\widehat{p}_{\hat{\mu}} = (p_\mu, p_4 = -mc)$; $G^{\mu\nu} = g^{\mu\nu}$; $G^{\hat{\mu}4} = G^{4\hat{\nu}} = 0$; $G^{44} = G_{44} = -1$.

This leads us to consider also the picture in the coordinate space and its extension to five dimensions. As in the previous case, we have a four-vector representing the space-time position of a particle

$$x^\mu = (ct, x, y, z)$$

and in view of the extension to general relativity

$$\mathrm{d}x^\mu = (c\mathrm{d}t, \mathrm{d}x, \mathrm{d}y, \mathrm{d}z) = (\mathrm{d}x^0, \mathrm{d}x^1, \mathrm{d}x^2, \mathrm{d}x^3). \tag{8}$$

The relativistic invariant is, in this case, the elementary interval ds, also expressed with the proper time τ of the particle

$$\mathrm{d}s^2 = \mathrm{d}x^\mu \mathrm{d}x_\mu = c^2\mathrm{d}t^2 - \mathrm{d}\,\overrightarrow{x}^2 = c^2\mathrm{d}\tau^2, \tag{9}$$

which is, as that was already the case for mass, equal to zero for light

$$\mathrm{d}s^2 = 0 \tag{10}$$

and this defines the usual light cone in space-time.

For massive particles proper time and interval are non-zero and eq. (9) defines again an hyperboloid. As in the energy-momentum picture we may enlarge our space-time with the additional dimension $s = c\tau$ (fig. 3)

$$\mathrm{d}\widehat{x}^{\widehat{\mu}} = (c\mathrm{d}t, \mathrm{d}x, \mathrm{d}y, \mathrm{d}z, c\mathrm{d}\tau) = (\mathrm{d}x^0, \mathrm{d}x^1, \mathrm{d}x^2, \mathrm{d}x^3, \mathrm{d}x^4) \tag{11}$$

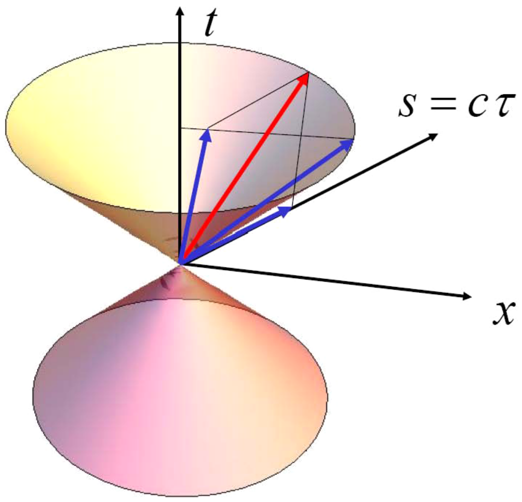

Fig. 3. – 5D coordinates.

and introduce a generalized light cone for massive particles([1])

$$d\sigma^2 = G_{\hat{\mu}\hat{\nu}} d\hat{x}^{\hat{\mu}} d\hat{x}^{\hat{\nu}} = c^2 dt^2 - d\vec{x}^2 - c^2 d\tau^2 = 0. \tag{12}$$

As pointed out in the case of mass, proper time is not defined by this equation from other coordinates but is rather a true evolution parameter representative of the internal evolution of the object. It coincides numerically with the time coordinate in the frame of the object through the relation

$$c d\tau = \sqrt{G_{00}} dx^0. \tag{13}$$

Finally, if we combine momenta and coordinates to form a mixed scalar product, we obtain a new relativistic invariant which is the differential of the action. In 4D

$$dS = -p_\mu dx^\mu \tag{14}$$

and in 5D we shall therefore introduce the superaction

$$\widehat{S} = -\int \widehat{p}_{\hat{\mu}} d\widehat{x}^{\hat{\mu}} \tag{15}$$

([1]) In this picture, antiparticles have a negative mass and propagate backwards on the fifth axis as first pointed out by Feynman. Still, their relativistic mass m^* is positive and hence they follow the same trajectories as particles in gravitational fields as we shall see from the equations of motion.

equivalent to

$$\widehat{p}_{\hat{\mu}} = -\frac{\partial \widehat{S}}{\partial \widehat{x}^{\hat{\mu}}} \quad \text{with} \quad \hat{\mu} = 0, 1, 2, 3, 4. \tag{16}$$

If this is substituted in

$$G^{\hat{\mu}\hat{\nu}} \widehat{p}_{\hat{\mu}} \widehat{p}_{\hat{\mu}} = 0, \tag{17}$$

we obtain the Hamilton-Jacobi equation in 5D

$$G^{\hat{\mu}\hat{\nu}} \partial_{\hat{\mu}} \widehat{S} \partial_{\hat{\nu}} \widehat{S} = 0, \tag{18}$$

which has the same form as the eikonal equation for light in 4D. It is already this striking analogy which pushed Louis de Broglie to identify action and the phase of a matter wave in the 4D case. We shall follow the same track for a quantum approach in our 5D case.

What is the link between the three previous invariants given above? As in optics, the direction of propagation of a particle is determined by the momentum vector tangent to the trajectory. The 5D momentum can therefore be written in the form

$$\widehat{p}^{\hat{\mu}} = \mathrm{d}\widehat{x}^{\hat{\mu}} / \mathrm{d}\lambda, \tag{19}$$

where λ is an affine parameter varying along the ray. This is consistent with the invariance of these quantities for uniform motion.

In 4D the canonical 4-momentum is

$$p_\mu = mc \frac{g_{\mu\nu} \mathrm{d}x^\nu}{\sqrt{g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu}} = mc g_{\mu\nu} u^\nu, \tag{20}$$

where $u^\nu = \mathrm{d}x^\nu / \mathrm{d}\tau$ is the normalized 4-velocity with $\mathrm{d}\tau = \sqrt{g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu}$ given by (9).

We observe that $\mathrm{d}\lambda$ can always be written as the ratio of a time to a mass

$$\mathrm{d}\lambda = \frac{\mathrm{d}\tau}{m} = \frac{\mathrm{d}t}{m^*} = \frac{\mathrm{d}\theta}{M} = \ldots, \tag{21}$$

where τ is the proper time of individual particles (*e.g.* atoms in a clock or in a molecule), t is the time coordinate of the composed object (clock, interferometer or molecule) and θ its proper time; m, m^*, M are respectively the mass, the relativistic mass of individual particles and their contribution to the scalar mass of the device or composed object.

In the usual paradigm of relativity, the time t is a coordinate variable and the proper time τ is taken as the evolution parameter to describe the motion of particles in space-time. In this presentation however, proper time is an independent coordinate describing the internal motion of massive particles, so that we shall rather chose the coordinate time as the evolution parameter. Another good reason for this choice is that, in order to describe an ensemble of atoms or of atom waves within a clock or an atom interferometer, it cannot be a good choice to use the proper time of a specific atom to describe the full device. We shall therefore write in 5D

$$\widehat{p}_{\hat{\mu}} = m^* G_{\hat{\mu}\hat{\nu}} \dot{\widehat{x}}^{\widehat{\nu}} = m^* \dot{\widehat{x}}_{\hat{\mu}} \tag{22}$$

expressed with the "relativistic mass"

$$m^* = m\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{mc}{\sqrt{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}} \tag{23}$$

and where the dot refers to derivation with respect to a "laboratory time" (identical to the proper time θ of the apparatus only in the absence of gravitation or inertial effects). With this choice $\dot{\widehat{x}}^0 = c$ and $\widehat{p}^0 = m^* c$. An alternate choice could be to take the proper time θ of the full device as the evolution parameter. In which case

$$c\mathrm{d}\theta = \sqrt{G_{00}}\mathrm{d}x^0 \quad \text{and} \quad M = m^* \sqrt{G_{00}}. \tag{24}$$

From

$$\mathrm{d}\sigma^2 = G_{\hat{\mu}\hat{\nu}} \mathrm{d}\widehat{x}^{\hat{\mu}} \mathrm{d}\widehat{x}^{\widehat{\nu}} = 0 \tag{25}$$

we infer in 5D

$$\mathrm{d}\widehat{S} = 0 \tag{26}$$

and in 4D

$$\mathrm{d}S = -p_\mu \mathrm{d}x^\mu = -mc^2 \mathrm{d}\tau. \tag{27}$$

In Appendix A of ref. [44], we generalize these relations to an object, such as a clock, a molecule, ..., composed of a number of subparticles and illustrate the origin of proper time as coming from the inner structure of the object.

3. – Generalization in the presence of gravitational and electromagnetic interactions

The previous 5D scheme can be extended to general relativity with a 4D metric tensor $g^{\mu\nu}$ and an electromagnetic 4-potential A_μ

$$g^{\mu\nu}(p_\mu - qA_\mu)(p_\nu - qA_\nu) = m^2c^2 \tag{28}$$

($q = -e$ for the electron).

We shall search for a metric tensor $G_{\hat{\mu}\hat{\nu}}$ for 5D such that the generalized interval given by

$$\mathrm{d}\sigma^2 = G_{\hat{\mu}\hat{\nu}}\mathrm{d}\widehat{x}^{\hat{\mu}}\mathrm{d}\widehat{x}^{\widehat{\nu}}$$

is an invariant.

Let us recall that, from the equivalence principle, the metric tensor $g^{\mu\nu}$ can be obtained from the Minkovski flat space-time tensor $\eta^{\mu\nu}$ using infinitesimal frame transformations from a locally inertial frame. Quite generally any infinitesimal coordinate transformation considered as a gauge transformation can be used to introduce a component of the gravito-inertial field. As an example, in 4D, the transformation (case of a rotation)

$$\begin{aligned} \mathrm{d}x'^i &= \mathrm{d}x^i + \alpha_0^i \mathrm{d}x^0, \\ \mathrm{d}x'^0 &= \mathrm{d}x^0 \end{aligned} \tag{29}$$

transforms the interval

$$\mathrm{d}s^2 = g'_{00}(\mathrm{d}x'^0)^2 + g'_{ij}\mathrm{d}x'^i\mathrm{d}x'^j \tag{30}$$

into

$$\mathrm{d}s^2 = g_{00}(\mathrm{d}x^0)^2 + 2g_{0i}\mathrm{d}x^0\mathrm{d}x^i + g_{ij}\mathrm{d}x^i\mathrm{d}x^j \tag{31}$$

with

$$g_{00} = g'_{00} + \alpha_0^i\alpha_0^j g'_{ij}, \tag{32}$$

$$g_{0j} = \alpha_0^i g'_{ij}, \tag{33}$$

$$g_{ij} = g'_{ij}, \tag{34}$$

$$g^{00} = g^{'00} = 1/g'_{00}, \tag{35}$$

$$g^{'ij} = 1/g'_{ij}. \tag{36}$$

Using

$$g_{ij}g^{i0} = -g^{00}g_{j0}, \tag{37}$$

we find

$$\alpha_0^i = -\frac{g^{i0}}{g^{00}}\,, \tag{38}$$

$$\alpha_0^i\alpha_0^j g'_{ij} = -\frac{g_{i0}g^{i0}}{g^{00}}\,. \tag{39}$$

In the case of rotation we recover the usual metric tensor in the rotating frame.

The action S becomes

$$S = -\int p'_\mu \mathrm{d}x'^\mu = -\int p'_0 \mathrm{d}x'^0 - \int p'_i \mathrm{d}x'^i \tag{40}$$

$$= -\int p'_0 \mathrm{d}x^0 - \int p'_i(\mathrm{d}x^i + \alpha^i_0 \mathrm{d}x^0)$$

$$S = -\int (p'_0 + p'_i \alpha^i_0)\mathrm{d}x^0 - \int p'_i \mathrm{d}x^i = -\int p_\mu \mathrm{d}x^\mu, \tag{41}$$

which gives the Sagnac phase as $\int (p_i g^{i0}/g^{00})\mathrm{d}x^0$.

The same approach can be used with the fifth dimension by introducing the gauge transformation

$$\begin{aligned} \mathrm{d}x'^4 &= \mathrm{d}x^4 + \beta^4_\mu \mathrm{d}\widehat{x}^\mu \\ \mathrm{d}\widehat{x}'^\mu &= \mathrm{d}\widehat{x}^\mu \end{aligned} \tag{42}$$

to generate the off-diagonal elements $G_{\mu 4}$

$$\mathrm{d}\sigma^2 = G_{44}(\mathrm{d}x^4)^2 + 2G_{44}\beta^4_\mu \mathrm{d}x^4 \mathrm{d}\widehat{x}^\mu + (g_{\mu\nu} + \beta^4_\mu \beta^4_\nu G_{44})\mathrm{d}\widehat{x}^\mu \mathrm{d}\widehat{x}^\nu$$

$$G_{44} = G'_{44}, \tag{43}$$

$$G_{\mu 4} = \beta^4_\mu G_{44}, \tag{44}$$

$$G_{\mu\nu} = g_{\mu\nu} + \beta^4_\mu \beta^4_\nu G_{44}. \tag{45}$$

The superaction $\widehat{S}$ given by (15) becomes

$$\widehat{S} = -\int \widehat{p}_{\hat{\mu}} \mathrm{d}\widehat{x}'^{\hat{\mu}} = -\int p_\mu \mathrm{d}\widehat{x}'^\mu - \int \widehat{p}_4 \mathrm{d}x'^4 \tag{46}$$

$$= -\int p_\mu \mathrm{d}\widehat{x}^\mu + \int mc(\mathrm{d}x^4 + \beta^4_\mu \mathrm{d}\widehat{x}^\mu) \tag{47}$$

$$\widehat{S} = -\int (p_\mu - mc\beta^4_\mu)\mathrm{d}\widehat{x}^\mu + \int mc^2 \mathrm{d}\tau, \tag{48}$$

which yields the Aharonov-Bohm phase if $mc\beta^4_\mu = qA_\mu$.

The metric tensor in five dimensions $G_{\mu\nu}$ is thus written as in Kaluza's theory to include the electromagnetic gauge field potential A_μ

$$G_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} G_{\mu\nu} & G_{\mu 4} \\ G_{4\nu} & G_{44} \end{pmatrix} = \begin{pmatrix} g_{\mu\nu} + \kappa^2 G_{44} A_\mu A_\nu & \kappa G_{44} A_\mu \\ \kappa G_{44} A_\nu & G_{44} \end{pmatrix}, \tag{49}$$

$$G^{\hat{\mu}\hat{\nu}} = \begin{pmatrix} G^{\mu\nu} & G^{\mu 4} \\ G^{4\nu} & G^{44} \end{pmatrix} = \begin{pmatrix} g^{\mu\nu} & -\kappa A^\mu \\ -\kappa A^\nu & G^{44} \end{pmatrix},$$

where κ is given by the gyromagnetic ratio of the object. This metric tensor is such that

$$(50)\quad G^{\hat{\mu}\widehat{\lambda}}G_{\widehat{\lambda}\hat{\nu}} = \begin{pmatrix} G^{\mu\lambda} & G^{\mu 4} \\ G^{4\lambda} & G^{44} \end{pmatrix}\begin{pmatrix} G_{\lambda\nu} & G_{\lambda 4} \\ G_{4\nu} & G_{44} \end{pmatrix} = \delta^{\hat{\mu}}_{\hat{\nu}}$$

$$= \begin{pmatrix} G^{\mu\lambda} & -\kappa A^{\mu} \\ -\kappa A^{\lambda} & G^{44} \end{pmatrix}\begin{pmatrix} g_{\lambda\nu} + \kappa^2 G_{44} A_\lambda A_\nu & +\kappa G_{44} A_\lambda \\ +\kappa G_{44} A_\nu & G_{44} \end{pmatrix}$$

$$= \begin{pmatrix} G^{\mu\lambda} g_{\lambda\nu} & \kappa G_{44} G^{\mu\lambda} A_\lambda - \kappa G_{44} A^{\mu} = 0 \\ -\kappa A^{\lambda}(g_{\lambda\nu} + \kappa^2 G_{44} A_\lambda A_\nu) + \kappa G_{44} G^{44} A_\nu & -\kappa^2 G_{44} A^{\lambda} A_\lambda + G^{44} G_{44} \end{pmatrix}$$

$$= \delta^{\hat{\mu}}_{\hat{\nu}}$$

which implies

$$(51)\quad \begin{aligned} G^{\mu\lambda} g_{\lambda\nu} &= \delta^{\mu}_{\nu} \\ G^{44} &= 1/G_{44} + \kappa^2 A^{\lambda} A_\lambda \end{aligned}$$

The equation

$$(52)\quad G^{\hat{\mu}\hat{\nu}}\widehat{p}_{\hat{\mu}}\widehat{p}_{\hat{\nu}} = 0,$$

with

$$(53)\quad \widehat{p}_{\hat{\mu}} = (p_\mu, -mc),$$

and $G_{44} = -1$ is therefore equivalent to eq. (28)

$$(54)\quad g^{\mu\nu}(p_\mu - qA_\mu)(p_\nu - qA_\nu) = m^2c^2.$$

Higher-order electromagnetic interactions are introduced via the multipolar expansion $p_\mu - qA_\mu + Q^{\lambda}F_{\mu\lambda}$, where dipole moments will become operators in the quantum description.

4. – Hamiltonian and Lagrangian: Parabolic approximation

In some cases it is convenient to assume that the energy E is close to a known value E_0 either because energy is conserved and remains equal to its initial value or because of a slow variation of parameters. This means that the usual hyperbolic dispersion curve is locally approximated by the parabola tangent to the hyperbola for the energy E_0. This approximation scheme applies to massive as well as to massless particles. We can then make use of the identity $E = \frac{E_0}{2} + \frac{E^2}{2E_0} + O(\varepsilon^2)$ valid to second-order in $\varepsilon = E - E_0$ (parabolic approximation).

Let us start with the exact formula

$$\widehat{p}^0 = \frac{\widehat{p}^0}{2} + \frac{(\widehat{p}^0)^2}{2\widehat{p}^0}\,, \tag{55}$$

in which $(\widehat{p}^0)^2$ is obtained from

$$0 = G^{\hat{\mu}\hat{\nu}}\widehat{p}_{\hat{\mu}}\widehat{p}_{\hat{\nu}} = G^{00}(\widehat{p}_0)^2 + 2G^{0\widehat{i}}\widehat{p}_0\widehat{p}_{\widehat{i}} + G^{\widehat{i}\widehat{j}}\widehat{p}_{\widehat{i}}\widehat{p}_{\widehat{j}} \tag{56}$$

$$= G^{00}(\widehat{p}_0 + \frac{G^{0\widehat{i}}}{G^{00}}\widehat{p}_{\widehat{i}})^2 + \left(G^{\widehat{i}\widehat{j}} - \frac{G^{0\widehat{i}}G^{0\widehat{j}}}{G^{00}}\right)\widehat{p}_{\widehat{i}}\widehat{p}_{\widehat{j}} \tag{57}$$

$$= \frac{1}{G^{00}}(\widehat{p}^0)^2 + \widehat{f}^{\widehat{i}\widehat{j}}\widehat{p}_{\widehat{i}}\widehat{p}_{\widehat{j}}, \tag{58}$$

i.e.:

$$(\widehat{p}^0)^2 = -G^{00}\widehat{f}^{\widehat{i}\widehat{j}}\widehat{p}_{\widehat{i}}\widehat{p}_{\widehat{j}}, \tag{59}$$

where

$$\widehat{f}^{\widehat{i}\widehat{j}} = G^{\widehat{i}\widehat{j}} - \frac{G^{0\widehat{i}}G^{0\widehat{j}}}{G^{00}} \tag{60}$$

is the 4D metric tensor, inverse of $G_{\widehat{i}\widehat{j}}$. Hence

$$\widehat{p}^0 = \frac{\widehat{p}^0}{2} - \frac{G^{00}\widehat{f}^{\widehat{i}\widehat{j}}\widehat{p}_{\widehat{i}}\widehat{p}_{\widehat{j}}}{2\widehat{p}^0} = G^{00}\widehat{p}_0 + G^{0\widehat{i}}\widehat{p}_{\widehat{i}} \tag{61}$$

and

$$\widehat{p}_0 = \frac{\widehat{p}^0}{2G^{00}} - \frac{\widehat{f}^{\widehat{i}\widehat{j}}\widehat{p}_{\widehat{i}}\widehat{p}_{\widehat{j}}}{2\widehat{p}^0} - \frac{G^{0\widehat{j}}\widehat{p}_{\widehat{j}}c}{G^{00}}\,, \quad \widehat{i},\widehat{j} = 1,2,3,4. \tag{62}$$

With the choice of time coordinate such that $\dot{\widehat{x}}^0 = c$ the Hamiltonian can be finally written as

$$H = \frac{m^*c^2}{2G^{00}} - \frac{\widehat{f}^{\widehat{i}\widehat{j}}\widehat{p}_{\widehat{i}}\widehat{p}_{\widehat{j}}}{2m^*} - \frac{G^{0\widehat{j}}\widehat{p}_{\widehat{j}}c}{G^{00}}\,, \quad \widehat{i},\widehat{j} = 1,2,3,4. \tag{63}$$

This expression is exact but requires the knowledge of the relativistic mass m^*. In the parabolic approximation this quantity will finally be approximated by its central value. From the previous exact expression of the Hamiltonian, the Lagrangian is recovered as

$$\widehat{L} = -\widehat{p}_{\hat{\mu}}\dot{\widehat{x}}^{\hat{\mu}} = -\frac{1}{2}m^*G_{\hat{\mu}\hat{\nu}}\dot{\widehat{x}}^{\hat{\mu}}\dot{\widehat{x}}^{\hat{\nu}}\,. \tag{64}$$

5. – Equations of motion

From this Lagrangian we may infer the following equations of motion:

$$\widehat{p}_{\hat{\mu}} = -\frac{\partial\widehat{L}}{\partial\dot{\widehat{x}}^{\hat{\mu}}} = m^* G_{\hat{\mu}\hat{\nu}}\dot{\widehat{x}}^{\hat{\nu}}, \tag{65}$$

i.e.

$$\dot{\widehat{x}}^{\widehat{i}} = \frac{\widehat{f}^{\widehat{i}\widehat{j}}\widehat{p}_{\widehat{j}}}{m^*} + \frac{G^{0\widehat{j}}c}{G^{00}} \tag{66}$$

and

$$\dot{\widehat{p}}_{\widehat{\lambda}} = \frac{1}{2}m^*(\partial_{\widehat{\lambda}}G_{\hat{\mu}\hat{\nu}})\dot{\widehat{x}}^{\hat{\mu}}\dot{\widehat{x}}^{\widehat{\nu}}, \tag{67}$$

or

$$\dot{\widehat{p}}^{\hat{\mu}} = \frac{1}{2}m^* G^{\hat{\mu}\widehat{\lambda}}(\partial_{\widehat{\lambda}}G_{\widehat{\kappa}\hat{\nu}} - 2\partial_{\widehat{\nu}}G_{\widehat{\lambda}\widehat{\kappa}})\dot{\widehat{x}}^{\widehat{\kappa}}\dot{\widehat{x}}^{\widehat{\nu}}. \tag{68}$$

These equations can be compared to those obtained either from the equation for geodesic lines in 5D obtained from $\delta d\sigma^2 = 0$ with

$$\mathrm{d}\sigma^2 = G_{\hat{\mu}\hat{\nu}}\mathrm{d}\widehat{x}^{\hat{\mu}}\mathrm{d}\widehat{x}^{\hat{\nu}}, \tag{69}$$

or from the condition

$$D\dot{\widehat{x}}^{\hat{\mu}} = 0. \tag{70}$$

We proceed as in 4D and find

$$\ddot{\widehat{x}}^{\hat{\mu}} + {}^{(5)}\Gamma^{\hat{\mu}}_{\widehat{\nu}\widehat{\lambda}}\dot{\widehat{x}}^{\widehat{\nu}}\dot{\widehat{x}}^{\widehat{\lambda}} = 0$$

with

$${}^{(5)}\Gamma^{\hat{\mu}}_{\widehat{\nu}\widehat{\lambda}}\dot{\widehat{x}}^{\widehat{\nu}}\dot{\widehat{x}}^{\widehat{\lambda}} = \frac{1}{2}G^{\hat{\mu}\widehat{\kappa}}(2\partial_{\widehat{\lambda}}G_{\widehat{\kappa}\widehat{\nu}} - \partial_{\widehat{\kappa}}G_{\widehat{\nu}\widehat{\lambda}})\dot{\widehat{x}}^{\widehat{\nu}}\dot{\widehat{x}}^{\widehat{\lambda}}. \tag{71}$$

We wish now to check that we recover the usual equations of motion in 4D when the metric is independent of the 5th coordinate

$$\ddot{\widehat{x}}^{\mu} +^{(5)}\Gamma^{\mu}_{\nu\lambda}\dot{\widehat{x}}^{\nu}\dot{\widehat{x}}^{\lambda} +^{(5)}\Gamma^{\mu}_{4\lambda}\dot{\widehat{x}}^{4}\dot{\widehat{x}}^{\lambda} +^{(5)}\Gamma^{\mu}_{\nu 4}\dot{\widehat{x}}^{4}\dot{\widehat{x}}^{\nu} +^{(5)}\Gamma^{\mu}_{44}\dot{\widehat{x}}^{4}\dot{\widehat{x}}^{4} = 0, \tag{72}$$

with

$$ {}^{(5)}\Gamma^{\mu}_{4\lambda} = \frac{G_{44}}{2}\kappa F^{\mu}_{\lambda}, \tag{73} $$

$$ {}^{(5)}\Gamma^{\mu}_{\nu 4} = \frac{G_{44}}{2}\kappa F^{\mu}_{\nu}, \tag{74} $$

$$ {}^{(5)}\Gamma^{\mu}_{44} = G^{\mu\nu}\partial_4 G_{\nu 4} = 0. \tag{75} $$

The Christoffel symbols in 4D and 5D are connected by

$$ {}^{(5)}\Gamma^{\mu}_{\nu\lambda} - {}^{(4)}\Gamma^{\mu}_{\nu\lambda} = \frac{\kappa^2}{2}(A_\nu F^{\mu}_{\lambda} + A_\lambda F^{\mu}_{\nu}). \tag{76} $$

Hence

$$ \ddot{\hat{x}}^{\mu} + {}^{(4)}\Gamma^{\mu}_{\nu\lambda}\dot{\hat{x}}^{\nu}\dot{\hat{x}}^{\lambda} + \frac{\kappa^2}{2}(A_\nu F^{\mu}_{\lambda} + A_\lambda F^{\mu}_{\nu})\dot{\hat{x}}^{\nu}\dot{\hat{x}}^{\lambda} = -G_{44}\dot{\hat{x}}^{4}\kappa F^{\mu}_{\lambda}\dot{\hat{x}}^{\lambda}; \tag{77} $$

using

$$ -G_{44}\dot{\hat{x}}^{4} = -\dot{\hat{x}}_4 + G_{4\lambda}\dot{\hat{x}}^{\hat{\lambda}}, \tag{78} $$

we recover the usual 4D equation of motion

$$ m^*(\ddot{\hat{x}}^{\mu} + {}^{(4)}\Gamma^{\mu}_{\nu\lambda}\dot{\hat{x}}^{\nu}\dot{\hat{x}}^{\lambda}) = qF^{\mu}_{\lambda}\dot{\hat{x}}^{\lambda}, \tag{79} $$

since

$$ m^*\dot{\hat{x}}_4 = \hat{p}_4 = -mc. \tag{80} $$

If we reintroduce the dependence in the fifth coordinate, we obtain the rate of mass change associated with the change of internal motion induced by an electromagnetic field

$$ \begin{aligned} \dot{\hat{p}}_4 &= \frac{1}{2}m^*(\partial_4 G_{\hat{\mu}\hat{\nu}})\dot{\hat{x}}^{\hat{\mu}}\dot{\hat{x}}^{\hat{\nu}} \\ &= m^*(\partial_4 G_{\mu 4})\dot{\hat{x}}^{\mu}\dot{\hat{x}}^{4} \end{aligned} \tag{81} $$

and similar expressions for the rate of energy-momentum changes induced by internal transitions. In the case of electric dipole transitions, the photon energy-momentum is exchanged at the Rabi frequency rate. However in this approximation we do not obtain the Rabi oscillations (pendellösung) which require to introduce two coupled modes and therefore a quantum treatment of their amplitudes.

6. – Wave equations for atom waves and wave packet propagation

The simplest approach to wave equations for atom waves ignores the spin associated with internal atomic states and uses a Klein-Gordon equation for each of these spinless states. This approach was extensively described in references [25, 33, 47]. The Klein-Gordon equation in 5D reads

$$\widehat{\Box}\varphi = G^{\hat{\mu}\hat{\nu}}\widehat{\nabla}_{\hat{\mu}}\widehat{\nabla}_{\hat{\nu}}\varphi = 0. \tag{82}$$

One can write a corresponding equation for the phase. This 5D eikonal equation replaces the Hamilton-Jacobi equation for massive particles in 4D (and the WKB approximation reduces to the eikonal approximation [22])

$$G^{\hat{\mu}\hat{\nu}}\partial_{\hat{\mu}}\phi\partial_{\hat{\nu}}\phi = 0, \tag{83}$$

whose solution is given by the 5D-superaction

$$h\phi = -\int \widehat{p}_{\hat{\mu}}\mathrm{d}\widehat{x}^{\hat{\mu}} = -\frac{E}{c}\left(\int c\mathrm{d}t - \int \frac{\mathrm{d}l^{(4)}}{\sqrt{G_{00}}} + \int \frac{G_{\widehat{j}0}}{G_{00}}\mathrm{d}x^{\widehat{j}}\right), \tag{84}$$

where the waves are assumed to be "monochromatic" (*i.e.* of constant energy E), where $\mathrm{d}l^{(4)}$ is the path length in 4-space given by the induced spatial metric $\hat{f}_{\widehat{ij}}$

$$\mathrm{d}l^{(4)} = \sqrt{-\hat{f}_{\widehat{ij}}\mathrm{d}x^{\widehat{i}}\mathrm{d}x^{\widehat{j}}},$$
$$\hat{f}_{\widehat{ij}} = G_{\widehat{ij}} - \frac{G_{0\widehat{i}}G_{0\widehat{j}}}{G_{00}}$$

and the corresponding wavelength is

$$\lambda^{(4)} = \frac{hc}{E}\sqrt{G_{00}},$$

which is the proper combination of de Broglie and de Broglie-Compton wavelengths (or wave vectors)

$$\left(\frac{1}{\lambda^{(4)}}\right)^2 = \frac{1}{G_{00}}\left(\frac{E}{hc}\right)^2 = \left(\frac{1}{\lambda_{\mathrm{dBC}}}\right)^2 + \left(\frac{1}{\lambda_{\mathrm{dB}}}\right)^2,$$

where, in the absence of interactions,

$$\lambda_{\mathrm{dBC}} = \frac{h}{mc} \quad \text{and} \quad \lambda_{\mathrm{dB}} = \frac{h}{p}.$$

There is presently no physical clock at the de Broglie-Compton frequency mc^2/h although it appears quite possible in the future through stimulated absorption and emission of photon pairs in the creation/annihilation process of electron-positron pairs. As we shall see later a real clock is generated at a Bohr frequency by a superposition of internal states and it oscillates at the difference of the two corresponding de Broglie-Compton frequencies on both sides of an interferometer.

The other components of $G_{\hat{\mu}\hat{\nu}}$ enter the expression of the "optical path length" as various contributions to a generalized index of refraction which describe various phase shifts: gravitational waves, Aharonov-Bohm, Aharonov-Casher.... The last term in (84) corresponds specifically to the Sagnac effect and to the shift induced by a scalar electric potential.

Many atom interferometers today work with atomic wave packets essentially non-monochromatic and for which the previous approach is not adapted. A relativistic Schroedinger-like equation can be written using the Hamiltonian derived above in the parabolic approximation. If furthermore this Hamiltonian is at most quadratic in position and momentum operators the best description of the propagation of the wave packets is provided by the extension to 5D of the ABCD formalism for Gaussian optics. The ABCD theorem states that a wave packet propagates along the classical trajectory with a phase factor given by the classical action [14,21,2,26,34]. In 5D the classical superaction cancels and the wave packet propagates in a way such that there is no dephasing along the classical trajectory followed by the wave packet center [47].

Ideally, one would require a wave equation for massive particles of arbitrary spin and mass and interacting with electromagnetic fields in the presence of gravito-inertial fields. This implies a lot of heavy mathematical formalism and we shall thus limit our considerations to spin-1/2 atoms in a two-level system. This was our approach in refs. [19,17] where coupled Dirac equations were written in curved space-time.

In the first approach, the following coupled Dirac equations are written for each level (with $\epsilon_{0123} = 1$)

$$\begin{aligned} &i\hbar\gamma^{\hat{\alpha}} e^{\mu}_{\hat{\alpha}}(\partial_\mu + \Gamma_\mu)\psi_b - m_b c\psi_b - (i\mu_b/2c)F_{\hat{\alpha}\hat{\beta}}\gamma^{\hat{\alpha}}\gamma^{\hat{\beta}}\psi_b - (i\mu^*_{ab}/4)\tilde{F}_{\hat{\alpha}\hat{\beta}}\gamma^{\hat{\alpha}}\gamma^{\hat{\beta}}\psi_a = 0, \\ &i\hbar\gamma^{\hat{\alpha}} e^{\mu}_{\hat{\alpha}}(\partial_\mu + \Gamma_\mu)\psi_a - m_a c\psi_a - (i\mu_a/2c)F_{\hat{\alpha}\hat{\beta}}\gamma^{\hat{\alpha}}\gamma^{\hat{\beta}}\psi_a - (i\mu_{ab}/4)\tilde{F}_{\hat{\alpha}\hat{\beta}}\gamma^{\hat{\alpha}}\gamma^{\hat{\beta}}\psi_b = 0, \end{aligned} \tag{85}$$

where $e^{\mu}_{\hat{\alpha}}$ is a tetrad or vierbein which defines a local inertial coordinate system in which a spinor can be introduced and $\Gamma_\mu = \frac{1}{8}[\gamma^{\hat{\alpha}}, \gamma^{\hat{\beta}}]e^{\nu}_{\hat{\alpha}}\nabla_\mu e_{\hat{\beta}\nu}$ is a spinorial connection. The tetrad field is obtained from the metric tensor by the condition

$$e^{\mu}_{\hat{\alpha}} e^{\nu}_{\hat{\beta}} g_{\mu\nu} = \eta_{\hat{\alpha}\hat{\beta}} = \mathrm{diag}(+---). \tag{86}$$

As pointed out in ref. [17], these equations can be cast together as a single equation for an eight-component isospinor and one can introduce a generalized covariant derivative and a generalized connection. Quite generally, a matter-wave interferometer can be viewed as a device to detect this connection. The previous equations may also be written with space-time dependent Fock-Ivanenko matrices $\gamma(x)$.

The relativistic phase shifts for Dirac particles have been derived and discussed at length from these equations [19, 17, 18, 21].

One can turn the previous equations into a 5D Dirac equation

$$\Lambda^{\hat{\mu}} D_{\hat{\mu}} \psi = 0$$

with new matrices

$$\begin{aligned}\Lambda^{\mu} &= \gamma^{\mu}\sigma^{0},\\ \Lambda^{4} &= (1+\kappa\gamma^{\mu}A_{\mu})\sigma^{0} + \zeta_{\alpha}\sigma^{\alpha}\gamma^{\mu}\gamma^{\nu}\tilde{F}_{\mu\nu},\end{aligned}$$

where the Pauli matrices refer to internal states. It is easy to generalize Volkov solution of Dirac equation to this 5D case. Another choice is to work in the interaction representation.

By iteration of this Dirac equation

$$\overline{\Lambda}^{\hat{\mu}} D_{\hat{\mu}} (\Lambda^{\hat{\nu}} D_{\hat{\nu}} \psi) = 0$$

with

$$\overline{\Lambda}^{\mu} = \Lambda^{\mu};\ \overline{\Lambda}^{4} = (-1+\kappa\gamma^{\mu}A_{\mu})\sigma^{0} + \zeta_{\alpha}\sigma^{\alpha}\gamma^{\mu}\gamma^{\nu}\tilde{F}_{\mu\nu},$$

we recover the Klein-Gordon equation

$$\widehat{\Box}\varphi = G^{\hat{\mu}\hat{\nu}}\widehat{\nabla}_{\hat{\mu}}\widehat{\nabla}_{\hat{\nu}}\varphi = 0. \tag{87}$$

7. – 5D expression of the phase shift

The total phase difference between both arms of an interferometer is usually calculated as the sum of three terms: the difference in the action integral along each path, the difference in the phases imprinted on the atom waves by the beam splitters and a contribution coming from the splitting of the wave packets at the exit of the interferometer [21, 25]. If α and β are the two branches of the interferometer:

$$\begin{aligned}\delta\phi(q) = &\sum_{j=1}^{N}[S_{\beta}(t_{j+1},t_j) - S_{\alpha}(t_{j+1},t_j)]/\hbar\\ &+\sum_{j=1}^{N}\left(\tilde{k}_{\beta j}q_{\beta j} - \tilde{k}_{\alpha j}q_{\alpha j}\right) - (\omega_{\beta j}-\omega_{\alpha j})\,t_j + (\varphi_{\beta j}-\varphi_{\alpha j})\\ &+[\tilde{p}_{\beta,D}(q-q_{\beta,D}) - \tilde{p}_{\alpha,D}(q-q_{\alpha,D})]/\hbar,\end{aligned} \tag{88}$$

where $S_{\alpha j} = S_{\alpha}(t_{j+1},t_j)$ and $S_{\beta j} = S_{\beta}(t_{j+1},t_j)$ are the action integrals along $\alpha(\beta)$ paths; $\hbar k_{\alpha j}(\hbar k_{\beta j})$ are the momenta transferred to the atoms by the j-th beam splitter along

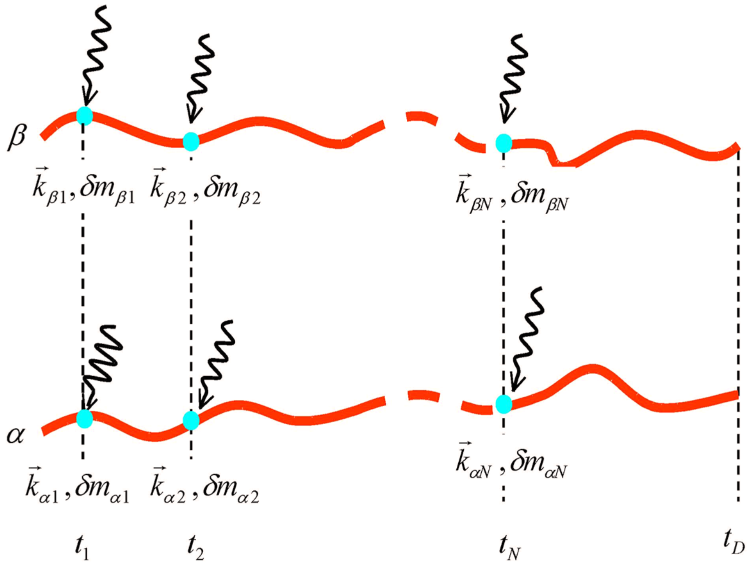

Fig. 4. – Time sequence of interactions on both arms of an interferometer displaying the 4D wave vector exchange at each beam splitter.

the $\alpha(\beta)$ arm; $q_{\alpha j}$ and $q_{\beta j}$ are the classical coordinates of the centers of the beam splitter/atom interactions; $\omega_{\alpha j}(\omega_{\beta j})$ are the angular frequencies of the *e.m.* waves; $\varphi_{\alpha j}(\varphi_{\beta j})$ are the fixed phases of the j-th beam splitters; D is the detection port (fig. 4).

With our new approach in 5D the action terms should be replaced by the phase jumps induced by the beam splitters along the fourth space coordinate $c\tau$

$$\sum_{j=1}^{N} c^2[\delta m_{\beta j}\tau_{\beta j} - \delta m_{\alpha j}\tau_{\alpha j}]/\hbar \tag{89}$$

in which $\delta m_{\beta j}(\delta m_{\alpha j})$ are the mass changes introduced by each splitter. To obtain this result we write the action terms as

$$\sum_{j=1}^{N} S_\beta(t_{j+1}, t_j) = \sum_{j=1}^{N} -c^2[m_{\beta j+1}\tau_{\beta j+1} - (m_{\beta j} + \delta m_{\beta j})\tau_{\beta j}] \tag{90}$$

with $m_{\beta N+1} = m_{\beta D}$ and $\tau_{\beta N+1} = \tau_{\beta D}$. We shift j by one unit for the first term

$$\sum_{j=1}^{N} S_\beta(t_{j+1}, t_j) = c^2 m_{\beta 1}\tau_{\beta 1} + \sum_{j=1}^{N} -c^2[m_{\beta j}\tau_{\beta j} - (m_{\beta j} + \delta m_{\beta j})\tau_{\beta j}] - c^2 m_{\beta D}\tau_{\beta D}. \tag{91}$$

We suppress the first term $c^2 m_{\beta 1}\tau_{\beta 1}$ and add a current term $c^2 m_{\beta D}\tau$ following the logic of a phase term analogous to the spatial terms (these terms are generally eliminated between both arms of the interferometer but they are indeed new phases arising in the 5D approach)

$$\sum_{j=1}^{N} S_\beta(t_{j+1}, t_j) \text{ is replaced by } \sum_{j=1}^{N} -c^2[m_{\beta j}\tau_{\beta j} - (m_{\beta j} + \delta m_{\beta j})\tau_{\beta j}] + c^2 m_{\beta D}(\tau - \tau_{\beta D})$$

$$= \sum_{j=1}^{N} c^2 \delta m_{\beta j}\tau_{\beta j} + c^2 m_{\beta D}(\tau - \tau_{\beta D}). \tag{92}$$

We have therefore in 5D

$$\delta\phi(q) = \sum_{j=1}^{N} \left(\tilde{k}_{\beta j} q_{\beta j} - \tilde{k}_{\alpha j} q_{\alpha j}\right) - (\omega_{\beta j} - \omega_{\alpha j})t_j + (\varphi_{\beta j} - \varphi_{\alpha j})$$
$$+ [\tilde{p}_{\beta,D}(q - q_{\beta,D}) - \tilde{p}_{\alpha,D}(q - q_{\alpha,D})]/\hbar \tag{93}$$

where $p_{\beta j}$, $\hbar k_{\beta j}$ and $q_{\beta j}$ have now an additional 4-component equal, respectively, to $m_{\beta j}c$, $\delta m_{\beta j}c$ and $c\tau_{\beta j}$. For Hermite-Gauss wave packets, this phase should be evaluated at the mid-point (mid-point theorem [2]) $q = (q_{\beta,D} + q_{\alpha,D})/2$. This mid-point phase shift is

$$\delta\phi((q_{\beta,D} + q_{\alpha,D})/2) = \sum_{j=1}^{N} \left(\tilde{k}_{\beta j} q_{\beta j} - \tilde{k}_{\alpha j} q_{\alpha j}\right) - (\omega_{\beta j} - \omega_{\alpha j})t_j + (\varphi_{\beta j} - \varphi_{\alpha j})$$
$$+ [(\tilde{p}_{\beta,D} + \tilde{p}_{\alpha,D})(q_{\alpha,D} - q_{\beta,D})/2]/\hbar. \tag{94}$$

If energy is conserved, we may use the conservation of the Lagrange invariant (derived from Stokes theorem)

$$(\tilde{p}_{\alpha j+1} + \tilde{p}_{\beta j+1})(q_{\beta j+1} - q_{\alpha j+1}) - \left[(\tilde{p}_{\alpha j} + \tilde{p}_{\beta j}) + \hbar\left(\tilde{k}_{\beta j} + \tilde{k}_{\alpha j}\right)\right](q_{\beta j} - q_{\alpha j}) = 0 \tag{95}$$

and obtain the 5D scalar product

$$\delta\phi((q_{\beta N+1} + q_{\alpha N+1})/2) = \sum_{j=1}^{N} \left[\frac{\tilde{k}_{\beta j} - \tilde{k}_{\alpha j}}{2}(q_{\beta j} + q_{\alpha j})\right] - (\omega_{\beta j} - \omega_{\alpha j})t_j + (\varphi_{\beta j} - \varphi_{\alpha j}) \tag{96}$$

(we have also assumed $q_{\beta 1} = q_{\alpha 1}$). If energy is not conserved, we may use instead the symplectic Lagrange-Helmholtz invariant in the quadratic approximation (Hamiltonian of degree 2 at most in position and momentum)

$$\frac{\tilde{p}_{\alpha j+1}}{m_\alpha^*}(q_{\beta j+1} - q_{\alpha j+1}) - \frac{\tilde{p}_{\alpha j}}{m_\alpha^*}(q_{\beta j} - q_{\alpha j}) = \frac{\tilde{p}_{\beta j+1}}{m_\beta^*}(q_{\alpha j+1} - q_{\beta j+1}) - \frac{\tilde{p}_{\beta j}}{m_\beta^*}(q_{\alpha j} - q_{\beta j}), \tag{97}$$

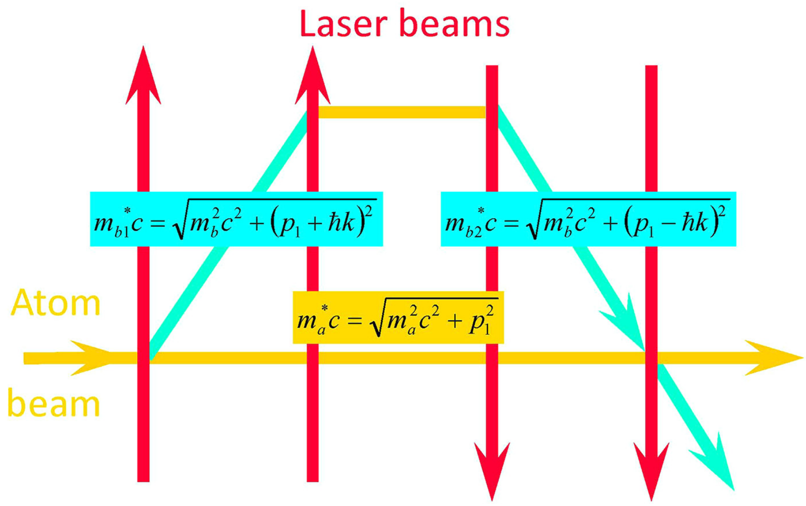

Fig. 5. – Bordé-Ramsey interferometer (higher-frequency recoil peak geometry) obtained with two counterpropagating pairs of copropagating travelling waves.

which reduces to the previous Lagrange invariant with a good approximation for small relative energy changes. This explains the cancellation of the action and of the mid-point phase shift in the usual 4D approach as emphasized in reference [47].

For illustration, let us apply the previous formulas to the Bordé-Ramsey interferometer [1, 16] represented in fig. 5.

Formula (94) gives for the mid-point 5D phase

$$
\begin{aligned}
(98)\qquad \delta\phi((\widehat{q}_{\beta 4} + \widehat{q}_{\alpha 4})/2) = {} & \vec{k}.\vec{q}_1 + (m_b - m_a)c^2\tau_1/\hbar - \omega t_1 \\
& -\vec{k}.\vec{q}_{\beta 2} + (-m_b + m_a)c^2\tau_{\beta 2}/\hbar + \omega t_2 \\
& -\vec{k}.\vec{q}_{\beta 3} + (m_b - m_a)c^2\tau_{\beta 3}/\hbar - \omega t_3 \\
& +\vec{k}.\vec{q}_{\beta 4} + (-m_b + m_a)c^2\tau_{\beta 4}/\hbar + \omega t_4 \\
& +\sum_{j=1}^{4}(\varphi_{\beta j} - \varphi_{\alpha j}) \\
& + \left[(\vec{p}_{\beta b4} + \vec{p}_{\alpha a4} + \hbar\vec{k}) \cdot (\vec{q}_{\alpha 4} - \vec{q}_{\beta 4})/2\right]/\hbar \\
& + [(m_b + m_a + m_a - m_b)(\tau_{\alpha 4} - \tau_{\beta 4})/2]\, c^2/\hbar .
\end{aligned}
$$

In the absence of gravito-inertial fields (*e.g.* in the inertial frame of the atoms):

$$t_2 = t_1 + T;\ \vec{q}_{\beta 2} = \vec{q}_1 + \frac{\left(\vec{p}_1 + \hbar\vec{k}\right)T}{m^*_{b1}};\ \tau_{\beta 2} = \tau_1 + \frac{m_b}{m^*_{b1}}T;\ \vec{p}_{\beta 2} = \vec{p}_1 + \hbar\vec{k}$$

$$t_3 = t_2 + T';\ \vec{q}_{\beta 3} = \vec{q}_1 + \frac{\left(\vec{p}_1 + \hbar\vec{k}\right)T}{m^*_{b1}} + \frac{\vec{p}_1 T'}{m^*_a};\ \tau_{\beta 3} = \tau_1 + \frac{m_b}{m^*_{b1}}T + \frac{m_a}{m^*_a}T';\ \vec{p}_{\beta b3} = \vec{p}_1$$

$$t_4 = t_3 + T;\ \vec{q}_{\beta 4} = \vec{q}_1 + \frac{\left(\vec{p}_1 + \hbar\vec{k}\right)T}{m^*_{b1}} + \frac{\left(\vec{p}_1 - \hbar\vec{k}\right)T}{m^*_{b2}} + \frac{\vec{p}_1 T'}{m^*_a};$$

$$\tau_{\beta 4} = \tau_1 + \frac{m_b}{m^*_{b1}}T + \frac{m_b}{m^*_{b2}}T + \frac{m_a}{m^*_a}T';\ \vec{p}_{\beta b4} = \vec{p}_1 - \hbar\vec{k}$$

and for the lower branch

$$\begin{aligned} &\vec{q}_{\alpha 2} = \vec{q}_1 + \frac{\vec{p}_1 T}{m^*_a};\ \tau_{\alpha 2} = \tau_1 + \frac{m_a}{m^*_a}T;\ \vec{p}_{\alpha 2} = \vec{p}_1, \\ &\vec{q}_{\alpha 3} = \vec{q}_1 + \frac{\vec{p}_1(T+T')}{m^*_a};\ \tau_{\alpha 3} = \tau_1 + \frac{m_a}{m^*_a}(T+T');\ \vec{p}_{\alpha 3} = \vec{p}_1, \\ &\vec{q}_{\alpha 4} = \vec{q}_1 + \frac{\vec{p}_1(2T+T')}{m^*_a};\ \tau_{\alpha 4} = \tau_1 + \frac{m_a}{m^*_a}(2T+T');\ \vec{p}_{\alpha 4} = \vec{p}_1. \end{aligned} \tag{99}$$

We see that the final positions and proper times differ on both arms by small quantities, owing to the relativistic differences of velocities on both arms

$$\begin{aligned} \vec{q}_{\beta 4} - \vec{q}_{\alpha 4} &= \left(\frac{1}{m^*_{b1}} + \frac{1}{m^*_{b2}} - \frac{2}{m^*_a}\right)\vec{p}_1 T + \left(\frac{1}{m^*_{b1}} - \frac{1}{m^*_{b2}}\right)\hbar\vec{k}T, \\ \tau_{\beta 4} - \tau_{\alpha 4} &= \left(\frac{m_b}{m^*_{b1}} + \frac{m_b}{m^*_{b2}} - \frac{2m_a}{m^*_a}\right)T. \end{aligned} \tag{100}$$

Finally

$$\delta\phi = [2\omega - (m^*_{b1} + m^*_{b2} - 2m^*_a)c^2/\hbar]T. \tag{101}$$

For each segment, we check the conservation of the symplectic invariant (Lagrange-Helmoltz), *e.g.*,

$$\left(\frac{\vec{p}_{\beta 2}}{m^*_{b1}} + \frac{\vec{p}_{\alpha 2}}{m^*_a}\right)\cdot(\vec{q}_{\alpha 2} - \vec{q}_{\beta 2})/2 + \left(\frac{m_b}{m^*_{b1}} + \frac{m_a}{m^*_a}\right)(\tau_{\alpha 2} - \tau_{\beta 2})c^2/2 = 0, \tag{102}$$

$$\begin{aligned} &\frac{\vec{p}_{\alpha 3} + \vec{p}_{\beta 3}}{m^*_a}\cdot(\vec{q}_{\beta 3} - \vec{q}_{\alpha 3}) - \frac{\vec{p}_{\alpha 2} + \vec{p}_{\beta 2}}{m^*_a}\cdot(\vec{q}_{\beta 2} - \vec{q}_{\alpha 2}) \\ &+\frac{2m_a}{m^*_a}(\tau_{\beta 3} - \tau_{\alpha 3})c^2/2 - \frac{2m_a}{m^*_a}(\tau_{\beta 2} - \tau_{\alpha 2})c^2/2 = 0, \end{aligned} \tag{103}$$

which correspond to a conserved Lagrange invariant only in the approximation of conserved energy along the arms of the interferometer. Within this approximation, we may use the approximate expression

$$\begin{aligned}
\delta\phi &= \vec{k}\cdot[\vec{q}_1-(\vec{q}_{\alpha 2}+\vec{q}_{\beta 2})/2-(\vec{q}_{\alpha 3}+\vec{q}_{\beta 3})/2+(\vec{q}_{\alpha 4}+\vec{q}_{\beta 4})/2] \\
&\quad +\omega_{ba}\left[\tau_1-(\tau_{\alpha 2}+\tau_{\beta 2})/2-(\tau_{\alpha 3}+\tau_{\beta 3})/2+(\tau_{\alpha 4}+\tau_{\beta 4})/2\right] \\
&\quad -\omega(t_1-t_2+t_3-t_4) \\
&= 2\omega t-\frac{\omega_{ba}T}{2}\left(2\frac{m_a}{m_a^*}+\frac{m_b}{m_{b1}^*}+\frac{m_b}{m_{b2}^*}\right) \\
&\quad -\frac{\hbar k^2T}{2}\left(\frac{1}{m_{b1}^*}+\frac{1}{m_{b2}^*}\right)+\frac{\vec{k}.\vec{p}_1T}{2}\left(\frac{1}{m_{b2}^*}-\frac{1}{m_{b1}^*}\right).
\end{aligned}\tag{104}$$

The same recoil and second-order Doppler corrections are obtained from this approximate formula to first-order but only expression (101) is exact. The second-term corresponds to the difference in proper times for the clock term (that we have called a quantum Langevin twin paradox in ref. [47]). Note that this clock term implies that the mass differs on both arms and does not correspond to a different clock on each arm as in the classical Langevin twin paradox. The coherent quantum superposition of both arms is essential to generate a clock. The remaining piece of the recoil shift comes from first-order Doppler shifts contained in the third term and originating from the spatial $\vec{q}$ part of the shift.

This recoil shift [6-8] is now the basis for accurate measurements of h/m and determinations of the fine structure constant α [55, 53, 54].

The same approach can be followed for the phase shift induced by gravito-inertial fields in photon echo interferometers [21]. For a gravimeter in the gravity field g and gradient γ

$$\begin{aligned}
\delta\phi &= -k((z_2+z_2')/2-z_1-z_1'+z_0) \\
&= -\frac{k}{\sqrt{\gamma}}\left[[\sinh(\sqrt{\gamma}(T+T'))-2\sinh(\sqrt{\gamma}T)]\left(V_0+\frac{\hbar k}{2m^*}\right)\right. \\
&\quad \left.+\sqrt{\gamma}[1+\cosh(\sqrt{\gamma}(T+T'))-2\cosh(\sqrt{\gamma}T)]\left(z_0-\frac{g}{\gamma}\right)\right],
\end{aligned}\tag{105}$$

which, to first-order in γ and for $T=T'$ reduces to

$$kgT^2+k\gamma T^2\left[\frac{7}{12}gT^2-(V_0+\frac{\hbar k}{2m^*})T-z_0\right].\tag{106}$$

The fact that the optical path difference in space, corresponding to the splitting of the end points, is exactly compensated by the action in 4D [21], is now understood as the

5D property that for massless particles equal paths are followed during equal times [22]. For a gyrometer [2]

$$\delta\phi_{Sagnac} = \sum_{j=1,4} \vec{k}_j \cdot \vec{r}_j + \vec{k}_4 \cdot \frac{(\vec{r}_4' - \vec{r}_4)}{2}\,, \tag{107}$$

which gives to first order the familiar expression proportional to energy

$$\delta\phi_{Sagnac} = \frac{2\vec{\Omega}.\vec{A}}{\hbar/m^*} \tag{108}$$

and more general formulas for time-dependent combinations of gravito-inertial fields [23, 24, 28].

For a gravitational wave detector [25]

$$\begin{aligned}\delta\phi = & -khV_0\xi T^2 \sin(\xi T + \varphi)\left[\sin^2(\xi T/2)/(\xi T/2)^2\right] \\ & -\frac{khq_0}{2}\left[\cos(2\xi T + \varphi) - 2\cos(\xi T + \varphi) + \cos\varphi\right] \\ & -khV_0 T\left[\cos(2\xi T + \varphi) - \cos(\xi T + \varphi)\right] + \varphi_0 - 2\varphi_1 + \varphi_2,\end{aligned} \tag{109}$$

where h and ξ are, respectively, the amplitude and the frequency of the gravitational wave.

The sensitivity comes essentially from the last two terms which can be interpreted as the response of the gravimeter to the phase modulation imposed on the laser beams by the gravitational wave. This method was already analyzed as early as 1983 in [50].

To obtain sensitivity from the action of the gravitational wave on the atom wave itself, we may rely on an additional recoil term in Bordé-Ramsey interferometers [19]

$$\delta\phi = -\frac{m^* c^2}{\hbar} T \left(\frac{\hbar k}{m^* c}\right)^2 h \cos(\xi T + \varphi)[\sin(\xi T)/(\xi T)], \tag{110}$$

since this term is proportional to the relativistic mass of the atoms provided that an efficient beam splitter is used to boost the ratio of momenta squared. As this was the case already for the Sagnac shift, there is a potential huge gain factor as one replaces the photon energy by the relativistic energy of the atom both described by $m^* c^2$.

8. – Conclusions

As a first conclusion, the motion of massive or massless particles in 5D follows a null geodesic just as it is the case for photons in 4D. The Lagrangian is proportional to the interval squared and both vanish for the real motion. This has the consequence that the phase, which is proportional to the 5D superaction, will also vanish between two points of the real trajectory of the particle. As a consequence the phase shift in atom interferometers results almost([2]) only from the phase jumps introduced by the beam splitters. The significance and status of the action in the 4D atomic phase are clarified and we offer an interpretation for the origin of compensations between action and spatial phase through the additional optical path along the fifth dimension.

The comparison between atom optics and photon optics becomes straightforward thanks to formulas valid for zero mass and at the same time one can easily discuss the possibilities of relativistic atom interferometry for gravitational wave detection.

A theoretical framework for the redefinition of the SI is provided by the connection between geometry, metric tensor and metrology and there are many implications for fundamental metrology [20].

Mass and proper time are entangled concepts which correspond to conjugate variables in classical mechanics and to non-commuting operators in quantum mechanics. Their respective units thus require a joint definition in which the unit of mass is defined from the mass difference of the two levels involved in the definition of the unit of time. A compatible *mise en pratique* requires to associate a quantum clock with a macroscopic mass through a phase measurement either by atom interferometry and atom counting or in the watt balance. The Avogadro number is then obtained directly from the measurement of the de Broglie-Compton frequency of the carbon atom.

The proper time acquires a status in Quantum Mechanics and we may now describe the quantum theory of atomic clocks in General Relativity from their internal properties [2].

Finally, temperature and proper time can be combined in a complex proper time variable in the theory of clocks. This accounts for thermal decoherence in atom interferometers. A generalized line shape for the usual Doppler broadening can be derived accordingly [36].

We have now a tool to perform the synthesis between atomic clocks and inertial sensors thanks to a single formula for the phase as well as a comparison of gravitational and electromagnetic (AB phase) effects on the atomic phase. The requirements to have a physical clock at the de Broglie-Compton frequency are clarified. No quantum observable oscillating at a given frequency can be generated unless a superposition of corresponding mass states is created between both arms of an interferometer [39, 41-43].

The consequences for the interpretation of quantum mechanics of a non-local hidden phase originating from the internal phase of a composed object have still to be explored in detail.

* * *

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([2]) *I.e.* up to a small recoil term.

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