Introduction to 5D optics for space-time sensors

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Summary. — A new framework is proposed to compare and unify photon and atom optics, which rests on the quantization of proper time. A common wave equation written in five dimensions reduces both cases to 5D-optics of massless particles. The ordinary methods of optics (eikonal equation, Kirchhoff integral, Lagrange invariant, Fermat principle, symplectic algebra and ABCD matrices....) are used to solve this equation in practical cases. The various phase shift cancellations, which occur in atom interferometers, and the quantum Langevin twin paradox for atoms, are then easily explained. A general phase-shift formula for interferometers is derived in five dimensions, which applies to clocks as well as to gravito-inertial sensors.

1. – Introduction

Space-time sensors include atomic clocks and gravito-inertial sensors [1]. Most space-time sensors today use either the interference of atom waves [2] or that of light beams. The same mathematical tools have been used to treat the propagation of both atoms and photons and to calculate phase shifts in interferometers: the WKB or eikonal approximation and the ABCD formalism [3]. One would like to go beyond the simple analogy and understand in depth what are the specific features brought by massive particles and what are the advantages of being able to modify the rest mass of the particles in interferometers [4, 5, 6]. The purpose of this course is to give a short introduction to generalizations of the WKB and ABCD formalisms for relativistic particles. A relativistic approach has the major advantage of providing a common framework for massive particles like atoms and massless ones like photons. A second step in this synthesis is to introduce mass as a quantum observable conjugate of proper time. Atom optics then becomes identical to photon optics in an extended (4+1)D space-time with 4 space dimensions. The optical path along the fourth dimension replaces the usual action phase factor in ordinary space-time. Finally, within the approximation of a slowly varying phase and amplitude of the field, the dispersion surface can be locally approximated by a tangent paraboloid and the dynamics is that of a non-relativistic massive particle in all cases. The corresponding propagator gives the generalized ABCD law. This will lead us to a general formula for the phase shifts in interferometry.

2. – Klein-Gordon equation for matter waves

Atoms in a given internal energy state can be treated as quanta of a matter-wave field with a rest mass M corresponding to this internal energy and a spin corresponding to the total angular momentum in that state. To take this spin into account one can use, for example, a Dirac [7, 8, 9], Proca or higher-spin wave equation. Here, for simplicity, we shall ignore this spin and start simply with the Klein-Gordon equation for the covariant wave amplitude of a scalar field.

The Lagrangian density for a complex scalar field φ is(1):

$$\mathcal{L}(x) = \hbar c\sqrt{-g}\left\{g^{\mu\nu}\partial_\mu\varphi^*\partial_\nu\varphi - \left[\frac{M^2c^2}{\hbar^2} + \xi R\right]\varphi^*\varphi\right\} \tag{1}$$

The Euler-Lagrange equation

$$\partial_\mu\left[\partial\mathcal{L}/\partial\left(\partial_\mu\varphi\right)\right] - \partial\mathcal{L}/\partial\varphi = 0 \tag{2}$$

yields the field equation:

$$\left[\Box + \frac{M^2c^2}{\hbar^2} + \xi R\right]\varphi = 0 \tag{3}$$

(1) In this equation g is the determinant of the metric tensor $g^{\mu\nu}$. The term ξR, where ξ is an arbitrary numerical factor and R the Ricci scalar curvature, is a possible coupling between the field φ and gravitation. This term is of interest for future investigations in atom interferometry but in the remainder of these lecture notes we shall assume mimimal coupling and take $\xi = 0$.

where the d'Alembertian is related to the curved space-time metric $g^{\mu\nu}$ by the usual expression:

$$\Box\varphi = g^{\mu\nu}\nabla_\mu\nabla_\nu\varphi = (-g)^{-1/2}\,\partial_\mu\left[(-g)^{1/2}\,g^{\mu\nu}\partial_\nu\varphi\right] \tag{4}$$

The canonical conjugate four-vector is:

$$\pi^\mu = \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\varphi\right)} = \hbar c\sqrt{-g}g^{\mu\nu}\partial_\nu\varphi^* \tag{5}$$

and the conserved four-current density:

$$j^\mu = -i\pi^\mu\varphi + c.c. = -i\hbar c\sqrt{-g}g^{\mu\nu}\varphi\partial_\nu\varphi^* + c.c. \tag{6}$$

The scalar product is thus:

$$(\varphi_1,\varphi_2) = -i\hbar\int_\Sigma g^{\mu\nu}\varphi_1\overleftrightarrow{\partial}_\nu\varphi_2^*\sqrt{-g}d\Sigma_\mu \tag{7}$$

An essential step in the formulation of the solution of the field equation is the derivation of its Green function $G(x,x^{'})$. Once this is achieved, boundary conditions and initial conditions can be introduced via a Kirchhoff-type representation of the solution (the magic rule of [10])

$$\varphi(x) = \int_\Sigma\Big(G(x,x^{'})\nabla'^\mu\varphi(x') - \varphi(x')\nabla'^\mu G(x,x^{'})\Big)\,d\Sigma_\mu + \int G(x,x^{'})\sqrt{-g}\rho(x')d^4x' \tag{8}$$

assuming that the values of the field and its normal derivative are known on a spacelike hypersurface Σ and where we have introduced a possible source term $\rho(x)$.

The Green function and propagator of the Klein-Gordon equation are well-known in a flat space-time but have no simple expression with an arbitrary metric tensor. In what follows, we shall explore various approximations to solve this problem, such as the WKB or the quadratic approximations.

3. – Introduction of the proper time coordinate

Previous equations assume that the massive particles have a unique well-defined mass M

$$\left[\Box + \frac{M^2c^2}{\hbar^2}\right]\varphi = 0 \tag{9}$$

In the case of a composite complex body, such as an atomic species, there is a rich spectrum of internal energies and hence masses. This spectrum is the spectrum of eigenvalues of the internal Hamiltonian H_0. Thus we need to generalize the Klein-Gordon equation to include this variable mass aspect. For eigenstates of the mass operator this is achieved thanks to the ersatz:

$$\varphi(x, c\tau) = \exp[i\frac{Mc^2}{\hbar}\ (\tau - \tau_0)]\varphi(x, Mc) \tag{10}$$

which obviously satisfies:

$$i\hbar\frac{\partial\varphi(x, c\tau)}{c\partial\tau} = -Mc\varphi(x, c\tau) \tag{11}$$

where τ is a new quantum variable that we shall identify with the proper time. Rest mass and proper time thus appear as conjugate variables, which are both Lorentz invariants. So that, in the general case, the Klein-Gordon equation (9) can be written:

$$\widehat{\Box}\varphi \equiv \left[\Box - \frac{1}{c^2}\frac{\partial^2}{\partial\tau^2}\right]\varphi = 0 \tag{12}$$

in which c is the only fundamental constant. Equation (12) is indeed the wave equation in a space with 4 spatial dimensions $x, y, z, c\tau$ [11]. We recover equation (9) in the case of a monomassic state for which the field oscillates at a single internal frequency Mc^2/h(2).

In the general case of a coherent superposition of mass states:

$$\varphi(x, c\tau) = \frac{1}{\sqrt{2\pi\hbar}}\int d(Mc)\ \exp[i\frac{Mc^2}{\hbar}\ (\tau - \tau_0)]\varphi(x, Mc) \tag{13}$$

or in Dirac notations:

$$< c\tau|\varphi >= \int d\,(Mc) < c\tau|Mc >< Mc|\varphi > \tag{14}$$

The conjugate variables are respectively: the proper interval

$$x^4 = -x_4 = c\tau \tag{15}$$

and the mass

$$p^4 = -p_4 = Mc \tag{16}$$

(2) An interesting analogy is that of a laser pointer illuminating a perpendicular screen. The light beam in (3+1)D satisfies the wave equation with the velocity c. The spot on the screen satisfies a Klein-Gordon equation in (2+1)D where the mass term is given by the wave vector component imposed by the laser cavity resonator in the third space dimension. In the subspace of the screen the velocity is reduced in the range from 0 to c.

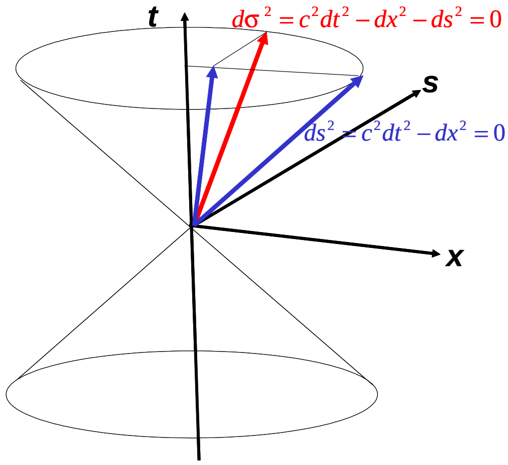

Fig. 1. – Elementary interval and generalized "light" cone in (4+1)D. Massive particles explore the additional space dimension $s = c\tau$ and have a velocity lower than c in space-time.

with the same relationship as between position and momentum:

$$(p_{op})_4 = -M_{op}c \tag{17}$$

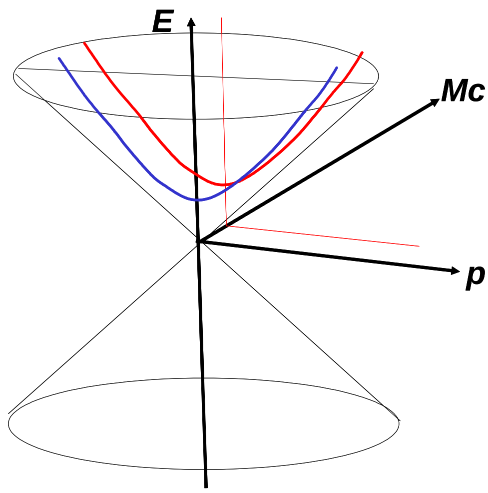

Fig. 2. – The dispersion surface is a cone in (4+1)D, which is projected in (3+1)D as a hyperboloid for each value of the mass.

so that:

$$(p_{op})_{\hat{\imath}}\, \varphi = i\hbar \partial_{\hat{\imath}} \varphi \quad \text{with} \quad \hat{\imath} = 1, 2, 3, 4 \tag{18}$$

(latin indices with a hat take the values 1,2,3,4 and greek ones the values 0,1,2,3,4).

Proper time (interval) and mass thus become non-commuting observables and this is well illustrated by the gedanken photon box experiment of Einstein and Bohr, which plays the role of the Heisenberg microscope for these two quantities.

In flat space-time, the propagator of equation (12) can be calculated, using standard techniques [10], either directly or from the Klein-Gordon propagator:

$$K^{(5)}\left(\vec{R}, \tau, T\right) = -\frac{1}{(2\pi)^2} \frac{1}{\left(c^2T^2 - R^2 - c^2\tau^2\right)^{3/2}} \tag{19}$$

Besides the mathematical difficulties in using this propagator, discussed in the 2D case in [10], it is not trivial to extend this result in the presence of arbitrary gravitational or inertial fields. We shall therefore turn to approximate methods: the WKB propagator in the case of constant energy and the ABCD propagator for time-dependent systems.

4. – WKB solution

We write the solution of the above field equations with a real amplitude and a real phase:

$$a \exp(i\phi) \tag{20}$$

The coupled equations satisfied by ϕ and a are respectively:

- a generalized Hamilton-Jacobi equation:

$$g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi = \frac{M^2 c^2}{\hbar^2} + \frac{\Box a}{a} \tag{21}$$

and

- a continuity equation:

$$\partial_\mu \left[(-g)^{1/2} g^{\mu\nu} \partial_\nu \phi \; a^2 \right] = 0 \tag{22}$$

which takes the familiar form:

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \vec{v}) = 0 \tag{23}$$

with: $\rho c = j^0 = -2\hbar c\, (-g)^{1/2} g^{0\nu} \partial_\nu \phi \; a^2$ and $\rho v^i = j^i = -2\hbar c\, (-g)^{1/2} g^{i\nu} \partial_\nu \phi \; a^2$.

If the quantum potential is neglected, the first equation becomes the usual Hamilton-Jacobi equation for massive particles in (3+1)D or as we shall see later an eikonal equation for massless particles in (4+1)D.

The usual Hamilton-Jacobi equation is satisfied by the classical action:

$$S = \int L dt = -\int p_\mu dx^\mu \tag{24}$$

since

$$\partial_0 S = -p_0 \quad \text{and} \quad \partial_i S = -p_i \tag{25}$$

and

$$g^{\mu\nu} p_\mu p_\nu = M^2 c^2 \tag{26}$$

From the relation between 4-momentum and 4-velocity (see the Appendix):

$$p^\mu = Mc \frac{\dot{x}^\mu}{\sqrt{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}} = M^* \dot{x}^\mu \tag{27}$$

one infers expressions for p_0^2 and p_i:

$$p_0^2 = g_{00} M^2 c^2 - g_{00} M^{*2} \gamma_{ij} \dot{x}^i \dot{x}^j \tag{28}$$

$$p_i = M^* \left(g_{ij} \dot{x}^j + g_{i0} \dot{x}^0 \right) = M^* \gamma_{ij} \dot{x}^j + \frac{g_{i0}}{g_{00}} p_0 \tag{29}$$

where

$$f_{ij} = g_{ij} - \frac{g_{0i} g_{0j}}{g_{00}} \tag{30}$$

is the 3D metric tensor.

From p_0^2 we get:

$$dt = M^* \sqrt{-\gamma_{ij} dx^i dx^j} / \sqrt{\frac{p_0^2}{g_{00}} - M^2 c^2} = M^* dl^{(3)} / \sqrt{\frac{p_0^2}{g_{00}} - M^2 c^2} \tag{31}$$

and the action is

$$S = -\int p_0 dx^0 - \int p_i dx^i = -p_0 c(t - t_0) + \int dl^{(3)} \sqrt{\frac{p_0^2}{g_{00}} - M^2 c^2} - p_0 \int \frac{g_{i0}}{g_{00}} dx^i \tag{32}$$

so that the de Broglie wavelength is:

$$\lambda_{dB} = \frac{h}{\sqrt{\frac{p_0^2}{g_{00}} - M^2c^2}} = \frac{h}{\sqrt{-f_{ij}p^ip^j}} \tag{33}$$

In the case of (4+1)D the total phase $\phi = S^{(4)}/\hbar$ satisfies the eikonal equation:

$$g^{\hat{\mu}\hat{\nu}}\partial_{\hat{\mu}}\phi\partial_{\hat{\nu}}\phi = 0 \text{ with } \hat{\mu},\hat{\nu} = 0,1,2,3,4 \tag{34}$$

Equations (28), (31) become:

$$p_0^2 \ = -g_{00}M^{*2}\dot{x}^4\dot{x}_4 - g_{00}M^{*2}f_{ij}\dot{x}^i\dot{x}^j \tag{35}$$

$$dt = \frac{M^*\sqrt{g_{00}}}{p_0}\sqrt{-f_{ij}dx^idx^j - dx_4dx^4} = \frac{M^*\sqrt{g_{00}}}{p_0}dl^{(4)} \tag{36}$$

and in the case of a constant energy the spatial part of the phase is now:

$$-\int\left(p_idx^i + p_4dx^4\right) = p_0\int\frac{\sqrt{-f_{ij}dx^idx^j - dx_4dx^4}}{\sqrt{g_{00}}} - p_0\int\frac{g_{i0}}{g_{00}}dx^i \tag{37}$$

The new optical path includes the path in the x^4 part of space. The de Broglie wavelength becomes:

$$\lambda_{dB}^{(4)} = \frac{h}{p_0} \tag{38}$$

and does not diverge any more for vanishing velocity.

The above expression of the path provides a generalization of Fermat's principle for the propagation of the waves associated with massive as well as massless particles:

$$\delta\int\left(\frac{dl^{(4)}}{\sqrt{g_{00}}} - \frac{g_{i0}}{g_{00}}dx^i\right) = 0 \tag{39}$$

where the integral to be varied is taken between two points along the ray in 4 space dimensions. This Fermat's principle proceeds from the existence of a Lagrange invariant $\oint p_{\hat{\imath}}dx^{\hat{\imath}}$ in this enlarged space, consequence of the fact that $p_{\hat{\imath}}$ is a gradient [15].

The photons propagate only in space-time where they have the maximum velocity c corrected by an index of refraction coming from g_{00} and also from f_{ij} (e.g. arising from the effect of gravitational waves). Massive particules propagate also in the additional space dimension $c\tau$ and may thus have a reduced velocity in ordinary space (Figure 1). They accumulate phase shifts along the four space coordinates. The phase along $c\tau$ is not affected by f_{ij} and by g_{i0}. Hence it will not be sensitive to gravitational waves or to rotation. This explains the reduced sensitivity of interferometers using non-relativistic particles to gravitational waves. Their enhanced sensitivity to rotation comes from the second term.

The phase shift in atom interferometers may now be understood in terms of optical paths only, just as this is the case for ordinary optics in (3+1)D space-time. For example, the phase cancellation, which occurs between the contributions of the action and that of the separation of the end points in space [13, 1, 16, 17], is easily understood from the fact that these points lie on the same wave front in the extended 4D-space.

Also the contribution to the recoil shift which originates from the action term in the phase [6, 14], has now an obvious interpretation as an optical path in the proper time dimension and constitutes a quantum realization of the Langevin twin paradox.

The continuity equation can also be integrated as in the book of Born and Wolf [15]. We can write it as:

$$a^2 \Box \phi + \partial^\mu \phi \partial_\mu a^2 = 0 \tag{40}$$

and if a^2 is time independent:

$$\partial^\mu \phi \partial_\mu a^2 = \partial^i \phi \partial_i a^2 = -\vec{\nabla} \phi \cdot \vec{\nabla} a^2 \tag{41}$$

Introducing the operator $\partial_\theta = \vec{\nabla} S \cdot \vec{\nabla}$ where θ is a parameter which specifies the position along the beam, we obtain :

$$\partial_\theta \ln a^2 = \Box S \tag{42}$$

From the Hamilton-Jacobi equation we get:

$$\partial_\theta S = -\partial^i S \partial_i S = \partial^0 S \partial_0 S - M^2 c^2 = g^{00} \left(\partial_0 S\right)^2 + g^{0i} \partial_0 S \partial_i S - M^2 c^2 \tag{43}$$

which gives

$$dS = \left(g^{00} p_0^2 \ + g^{0i} p_0 p_i - M^2 c^2\right) d\theta$$

When this is compared to 32:

$$dS = \sqrt{\frac{p_0^2}{g_{00}} - M^2 c^2 dl^{(3)}} - p_0 \frac{g_{i0}}{g_{00}} dx^i \tag{44}$$

one infers that

$$d\theta = \frac{dl^{(3)}}{\sqrt{\frac{p_0^2}{g_{00}} - M^2c^2}} \tag{45}$$

From which we can integrate (42) along the ray:

$$a^2 = a_0^2 \exp\left[\int \frac{\Box S^{(3)} dl^{(3)}}{\sqrt{\frac{p_0^2}{g_{00}} - M^2c^2}}\right]$$

for (3+1)D. Note that the d'Alembertian reduces to the Laplacian (with a minus sign) in the case of uniform rotation.

When this calculation is repeated without mass but with hatted indices for (4+1)D one gets:

$$d\hat{\theta} = \frac{\sqrt{g_{00}}dl^{(4)}}{p_0} = \frac{dt}{M^*} \tag{46}$$

and

$$a^2 = a_0^2 \exp\left[\int \sqrt{g_{00}}\frac{\widehat{\Box} S^{(4)}}{p_0} dl^{(4)}\right] \tag{47}$$

This formula solves the problem of finding the prefactor of the WKB propagator.

5. – Hamiltonian and Lagrangian expressions in the parabolic approximation

5˙1. *Classical derivation.* – First, we shall follow a simple track, starting with the approximations on classical formulas and turn later to quantum mechanics. From the classical relation between the 4-momentum p_μ, the metric tensor $g^{\mu\nu}$(with signature $+---$) and the rest mass of a particle M:

$$g^{\mu\nu}p_\mu p_\nu = M^2c^2 \tag{48}$$

and the relation between the covariant component p_0c (energy) and the contravariant one p^0c (relativistic mass times c^2) :

$$p^0c = g^{00}p_0c + g^{0i}p_ic \tag{49}$$

we obtain

$$\left(p^0\right)^2 = g^{00}(M^2c^2 - p_if^{ij}p_j) \tag{50}$$

where

$$f^{ij} = g^{ij} - \frac{g^{0i}g^{0j}}{g^{00}} \tag{51}$$

is the 3D metric tensor.

The component p^0c is related to the relativistic mass M^* (see the Appendix) through

$$p^0c = M^*c^2 \tag{52}$$

and can be written

$$p^0c = \frac{M^*c^2}{2} + \frac{\left(M^2c^2 - p_i f^{ij} p_j\right) g^{00}}{2M^*} \tag{53}$$

If M^*c^2 is approximated by a known prescribed function of time, this formula remains valid to second-order (parabolic approximation) since:

$$x = \frac{x_0}{2} + \frac{x^2}{2x_0} + O\left((x - x_0)^2\right)$$

and the Hamiltonian p_0c can be written as :

$$H = \frac{M^*c^2}{2g^{00}} + \frac{M^2c^2}{2M^*} - \frac{1}{2M^*} p_i f^{ij} p_j - \frac{g^{0i}}{g^{00}} p_i c \tag{54}$$

$$i, j = 1, 2, 3 \tag{55}$$

The Lagrangian is then:

$$\begin{aligned} L &= -p_i \dot{x}^i - H \\ &= -\frac{c^2}{2}\left(M^* g_{00} + \frac{M^2}{M^*}\right) - \frac{1}{2} M^* \dot{x}^i g_{ij} \dot{x}^j \\ &\quad -M^* c g_{0i} \dot{x}^i - \frac{1}{2} M^* c^2 \frac{g^{0i} g_{0i}}{g^{00}} \end{aligned} \tag{56}$$

In some cases it may be more convenient to assume that the energy E is close to a known value E_0 either because energy is conserved and remains equal to its initial value or because of a slow variation of parameters. We can again make use of the identity: $E = \frac{E_0}{2} + \frac{E^2}{2E_0} + O(\varepsilon^2)$ valid to second-order in $\varepsilon = E - E_0$ with either:

$$E^2 = p_0^2 c^2 = \frac{c^2}{g^{00}} \left(M^2c^2 - p_i g^{ij} p_j - 2p_0 g^{0i} p_i\right) \tag{57}$$

or

$$E^2 = p_0^2 c^2 = g_{00} M^2 c^4 - g_{00} c^2 p^i f_{ij} p^j \tag{58}$$

In the parabolic approximation, the Hamiltonian can then be approximated by:

$$H = \frac{E_0}{2} + \frac{M^2 c^4}{2E_0 g^{00}} - \frac{c^2}{2E_0 g^{00}} \left(p_i g^{ij} p_j + 2p_0 g^{0i} p_i\right) \tag{59}$$

$$\simeq \frac{E_0}{2} + \frac{M^2 c^4}{2E_0 g^{00}} - \frac{c^2}{2E_0 g^{00}} p_i g^{ij} p_j + \frac{g^{0i}}{g^{00}} p_i c \tag{60}$$

or by:

$$H = \frac{E_0}{2} + g_{00} \frac{M^2 c^4}{2E_0} - g_{00} c^2 \frac{p^i f_{ij} p^j}{2E_0} \tag{61}$$

This means that the usual hyperbolic dispersion curve is locally approximated by the parabola tangent to the hyperbola for the energy E_0 [18]. This approximation scheme applies to massive as well as to massless particles (For example in the case of quasi-monochromatic light $M = 0$ and $E_0 = \hbar\omega$ [3]). The non-relativistic limit is obtained for $M^* \rightarrow M\sqrt{g^{00}}$or $E_0 \rightarrow Mc^2/\sqrt{g^{00}}$. All these forms of the Hamiltonian can be shown to be equivalent thanks to (49).

From the above Hamiltonians we can deduce a Schroedinger-like equation:

$$i\hbar \frac{\partial \varphi}{\partial t} = \left[\frac{M^* c^2}{2g^{00}} + \frac{M^2 c^2}{2M^*} + \frac{\hbar^2}{2M^*} \partial_i f^{ij} \partial_j - i\hbar c \frac{g^{0i}}{g^{00}} \partial_i\right] \varphi \tag{62}$$

which is identical to a Schroedinger equation for a non-relativistic particle of mass $M^*(t)$ classical relativistic mass and with a shift in the rest mass. As we shall see in the next paragraph, this equation can also be derived directly from Klein-Gordon equation by a procedure analogous to that of H. Feshbach and F. Villars [12] in which the rest mass is replaced by the relativistic mass.

5˙2. *Quantum mechanical derivation.* – We generalize the procedure introduced by H. Feshbach and F. Villars [12] and a two-component wave function Ψ is defined through the combinations:

$$\Psi = \begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \varphi + \alpha\chi \\ \varphi - \alpha\chi \end{pmatrix}$$

where α is a function of time only and where:

$$\chi = i\pi^{0*} = i\hbar c \sqrt{-g} g^{0\nu} \partial_\nu \varphi \tag{63}$$

The Klein-Gordon equation is equivalent to the set of two equations:

$$i\hbar\partial_t\varphi = -i\hbar c\frac{g^{0k}}{g^{00}}\partial_k\varphi + \frac{1}{\sqrt{-g}g^{00}}\chi \tag{64}$$

$$i\hbar\partial_t\chi = M^2c^4\sqrt{-g}\varphi + \hbar^2c^2\partial_i\left(\sqrt{-g}f^{ij}\partial_j\right)\varphi - i\hbar c\partial_k\left(\frac{g^{0k}}{g^{00}}\chi\right) \tag{65}$$

where again

$$f^{ij} = g^{ij} - \frac{g^{0i}g^{0j}}{g^{00}} \tag{66}$$

From which

$$\begin{aligned} i\hbar\partial_t\Psi = {} & \frac{(\sigma_3 + i\sigma_2)\,\alpha}{2}\left[M^2c^4\sqrt{-g} + \hbar^2c^2\partial_i\left(\sqrt{-g}f^{ij}\partial_j\right)\right]\Psi \\ & + \frac{(\sigma_3 - i\sigma_2)}{2\alpha\sqrt{-g}g^{00}}\Psi - i\hbar c\frac{g^{0k}}{g^{00}}\partial_k\Psi \\ & - \left[\frac{i\hbar c}{2}(\sigma_0 - \sigma_1)\,\partial_k\left(\frac{g^{0k}}{g^{00}}\right)\right]\Psi \\ & + \left[\frac{i\hbar}{2\alpha}(\sigma_0 - \sigma_1)\,\partial_t\alpha\right]\Psi \end{aligned} \tag{67}$$

where $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices.

One could proceed with Foldy-Wouthuysen transformations, but we may also chose α in order to decouple as much as possible large and small components. This requires that along the classical trajectory one should have, on the average:

$$\alpha = \frac{1}{M^*c^2 < \sqrt{-g} >} \tag{68}$$

and for the large component we recover the Hamiltonian derived classically if the small corrections terms required for hermiticity and the term implying $\partial_t\alpha/\alpha$ are ignored.

6. – Schroedinger-like equation in (4+1)D

In (4+1) dimensions, the Hamiltonian becomes:

$$H = \frac{M^*c^2}{2g^{00}} - \frac{1}{2M^*}p_{\hat{\imath}}f^{\hat{\imath}\hat{\jmath}}p_{\hat{\jmath}} - \frac{g^{0\hat{\imath}}}{g^{00}}p_{\hat{\imath}}c \tag{69}$$

$$\hat{\imath}, \hat{\jmath} = 1, 2, 3, 4 \tag{70}$$

where we have introduced an extended metric tensor $g^{\hat{\mu}\hat{\nu}}$ (greek indices with a hat have integer values from 0 to 4: $\hat{\mu}, \hat{\nu} = 0, 1, 2, 3, 4$ and latin ones from 1 to 4) such that $g^{44} = -1$. Note that the components $g^{4\mu}$ could be used to represent electromagnetic interactions as in Kaluza-Klein theory.

In (4+1)D, with:

$$(p_{op})_{\hat{\imath}}\, \varphi = i\hbar \partial_{\hat{\imath}} \varphi \tag{71}$$

the Schroedinger equation becomes:

$$i\hbar \frac{\partial \varphi}{\partial t} = \left[\frac{M^* c^2}{2g^{00}} + \frac{\hbar^2}{2M^*} \partial_{\hat{\imath}} f^{\hat{\imath}\hat{\jmath}} \partial_{\hat{\jmath}} - i\hbar c \frac{g^{0\hat{\imath}}}{g^{00}} \partial_{\hat{\imath}} \right] \varphi \tag{72}$$

7. – Weak-field approximation

In the weak-field approximation the space-time metric tensor takes the form

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad\qquad |h_{\mu\nu}| << 1. \tag{73}$$

where $\eta_{\mu\nu}$ is Minkovski metric tensor:

$$\eta_{\mu\nu} = (1, -1, -1, -1) \tag{74}$$

and the $h_{\mu\nu}$'s are considered as first-order quantities.

In (4+1)D the Hamiltonian is:

$$H = \frac{M^* c^2}{2} \left(1 + h^{00}\right) - \frac{1}{2M^*} p_{\hat{\imath}} \eta^{\hat{\imath}\hat{\jmath}} p_{\hat{\jmath}} + \frac{1}{2M^*} p_i h^{ij} p_j + h^{0i} p_i c \tag{75}$$

$$\hat{\imath}, \hat{\jmath} = 1, 2, 3, 4 \qquad \eta_{\hat{\mu}\hat{\nu}} = (1, -1, -1, -1, -1) \tag{76}$$

and the field equation can be written as an ordinary Schroedinger equation in flat space-time with an additional spatial dimension $c\tau$:

$$i\hbar \frac{\partial \varphi}{\partial t} = \frac{M^* c^2}{2} \left(1 + h^{00}\right) \varphi - \frac{\hbar^2}{2M^*} \left(\nabla^2 \varphi + \frac{1}{c^2} \frac{\partial^2}{\partial \tau^2} \right) - \frac{\hbar^2}{2M^*} \partial_i h^{ij} \partial_j \varphi + i\hbar c h^{0i} \partial_i \varphi \tag{77}$$

In most cases of interest for atom interferometry, the external motion Hamiltonian can be expressed as a quadratic polynomial of both momentum and position. In this quadratic limit:

$$h^{00} = \frac{2g_i x^i}{c^2} - \frac{x^i \gamma_{ij} x^j}{c^2} \tag{78}$$

$$ch^{0i} = -f^i + \alpha^i_j x^j \tag{79}$$

$$h^{ij} = \eta^{ij} + \beta^{ij} \tag{80}$$

The (4+1)D Hamiltonian becomes:

$$\begin{aligned} H = & \frac{M^* c^2}{2} + \frac{1}{2}\vec{p}_{op}.\overset{\Rightarrow}{\alpha}(t).\vec{q}_{op} + \frac{1}{2M^*}\vec{p}_{op}.\overset{\Rightarrow}{\beta}(t).\vec{p}_{op} \\ & -\frac{1}{2}\vec{q}_{op}.\overset{\Rightarrow}{\delta}(t).\vec{p}_{op} - \frac{M^*}{2}\vec{q}_{op}.\overset{\Rightarrow}{\gamma}(t).\vec{q}_{op} \\ & +\overrightarrow{f}(t).\vec{p}_{op} - M^*\overrightarrow{g}(t).\vec{q}_{op} \end{aligned} \tag{81}$$

where vectors and tensors are now defined in a four dimensional space $\tilde{q} = (x, y, z, c\tau), \tilde{p} = (p_x, p_y, p_z, Mc)$ with $\beta_{44} = 1$.

It is convenient to introduce a new notation:

$$\underline{\overset{\Rightarrow}{\beta}}(t) = \overset{\Rightarrow}{\beta}(t)/M^*, \underline{\overset{\Rightarrow}{\gamma}}(t) = M^*\overset{\Rightarrow}{\gamma}(t) \ , \underline{\overrightarrow{g}}(t) = M^*\overrightarrow{g}(t) \tag{82}$$

with which the wave equation now reads:

$$\begin{aligned} i\hbar\frac{\partial\varphi}{\partial t} = & [\frac{M^* c^2}{2} + \frac{1}{2}\vec{p}_{op}.\overset{\Rightarrow}{\alpha}(t).\vec{q}_{op} + \frac{1}{2}\vec{p}_{op}.\underline{\overset{\Rightarrow}{\beta}}(t).\vec{p}_{op} \\ & -\frac{1}{2}\vec{q}_{op}.\overset{\Rightarrow}{\delta}(t).\vec{p}_{op} - \frac{1}{2}\vec{q}_{op}.\underline{\overset{\Rightarrow}{\gamma}}(t).\vec{q}_{op} \\ & +\overrightarrow{f}(t).\vec{p}_{op} - \underline{\overrightarrow{g}}(t).\vec{q}_{op}]\varphi \end{aligned} \tag{83}$$

8. – Propagator and ABCD law

The propagator of the previous equation with four space dimensions can be derived as what was done in [1] for three space dimensions:

$$\begin{aligned} \mathcal{K}(q, q', t, t') = & \left(\frac{1}{2\pi i\hbar}\right)^2 |\det \underline{B}|^{-1/2} \\ & \exp\left[\left(\frac{i}{2\hbar}\right)\left[\tilde{q}D\underline{B}^{-1}\ q - 2\tilde{q}\ \widetilde{\underline{B}^{-1}}\ q' + \tilde{q}'\ \underline{B}^{-1}A\ q'\right]\right] \end{aligned} \tag{84}$$

with $\tilde{q} = (x, y, z, c\tau), \tilde{p} = (p_x, p_y, p_z, Mc)$ and this leads to a new the ABCD theorem which gives the propagation law for Hermite-Gauss wave packets:

$$wave\_packet(q,t) = \exp\left[\frac{i\tilde{p}_c(t)(q-q_c(t))}{\hbar}\right] F(q-q_c(t), p_c(t), X(t), Y(t)) \tag{85}$$

where the phase factor usually associated with the classical action has disappeared and is cancelled by a term coming from the motion of the wave packet in the fourth space coordinate $c\tau$:

$$\begin{aligned}&\int_{-\infty}^{+\infty} d\tau' \left(\frac{1}{2\pi i\hbar \underline{B}_{44}}\right)^{1/2} \exp\left[(i/2\hbar)\underline{B}_{44}^{-1}c^2(\tau-\tau')^2\right] \\ &\exp\left[iMc^2(\tau'-\tau_c(t_0))/\hbar\right] F\left(\tau'-\tau_c(t_0), X_0, Y_0\right) \\ &= \exp\left[\frac{iM^2c^2}{2\hbar}\underline{B}_{44}\right] \exp\left[\frac{iMc^2}{\hbar}\left(\tau-\tau_c(t)\right)\right] F(\tau-\tau_c(t), X_\tau(t), Y_\tau(t))\end{aligned} \tag{86}$$

where F is the generating function of Hermite-Gauss wave packets [1]. The center of the wave packet follows a classical law:

$$q_c(t) = A\,(t,t_0)\, q_c(t_0) + \underline{B}\,(t,t_0)\, p_c(t_0) + \xi\,(t,t_0) \tag{87}$$

$$p_c(t) = \underline{C}\,(t,t_0)\, q_c(t_0) + D\,(t,t_0)\, p_c(t_0) + \underline{\phi}\,(t,t_0) \tag{88}$$

including the classical relation between proper time and time coordinate (obtained by integration of A.14):

$$\tau_c(t) = \tau_c(t_0) + \underline{B}_{44}\,(t,t_0)\, M = \tau_c(t_0) + \int_{t_0}^{t} \frac{M}{M^*(t')} dt' \tag{89}$$

We know from Ehrenfest theorem, that the motion of the wave packet center is indeed obtained in this case from classical equations. The equations satisfied by the $ABCD$ matrices can be derived either from the Hamilton-Jacobi equation (see [13]) or from Hamilton's equations [18]. For the previous Hamiltonian, Hamilton's equations can be written as an equation for the two-component vector:

$$\underline{\chi} = \begin{pmatrix} q \\ p \end{pmatrix} \tag{90}$$

as:

$$\frac{d\underline{\chi}}{dt} = \begin{pmatrix} \frac{dH}{dp} \\ -\frac{dH}{dq} \end{pmatrix} = \underline{\Gamma}\,(t)\,\underline{\chi} + \underline{\Phi}\,(t) \tag{91}$$

where

$$\underline{\Gamma}(t) = \begin{pmatrix} \alpha(t) & \underline{\beta}(t) \\ \underline{\gamma}(t) & \delta(t) \end{pmatrix} \tag{92}$$

is a time-dependent 8x8 matrix and where:

$$\underline{\Phi}(t) = \begin{pmatrix} f(t) \\ \underline{g}(t) \end{pmatrix} \tag{93}$$

The integral of Hamilton's equation can thus be written as:

$$\underline{\chi}(t) = \begin{pmatrix} A(t,t_0) & \underline{B}(t,t_0) \\ \underline{C}(t,t_0) & D(t,t_0) \end{pmatrix} \underline{\chi}(t_0) + \begin{pmatrix} \xi(t,t_0) \\ \underline{\phi}(t,t_0) \end{pmatrix} \tag{94}$$

where

$$\underline{\mathcal{M}}(t,t_0) = \begin{pmatrix} A(t,t_0) & \underline{B}(t,t_0) \\ \underline{C}(t,t_0) & D(t,t_0) \end{pmatrix} = \mathcal{T} \exp \left[\int_{t_0}^{t} \begin{pmatrix} \alpha(t') & \underline{\beta}(t') \\ \underline{\gamma}(t') & \delta(t') \end{pmatrix} dt' \right] \tag{95}$$

where $\mathcal{T}$ is a time-ordering operator and where:

$$\begin{pmatrix} \xi(t,t_0) \\ \underline{\phi}(t,t_0) \end{pmatrix} = \int_{t_0}^{t} \underline{\mathcal{M}}(t,t') \, \underline{\Phi}(t') \, dt' \tag{96}$$

9. – Phase-shift formula for atom interferometers

The total phase difference between both arms of an interferometer is usually calculated as the sum of three terms: the difference in the action integral along each path, the difference in the phases imprinted on the atom waves by the beam splitters and a contribution coming from the splitting of the wave packets at the exit of the interferometer [13, 18]. If α and β are the two branches of the interferometer:

$$\begin{aligned} \delta\phi(q) = & \sum_{j=1}^{N} \left[S_\beta(t_{j+1}, t_j) - S_\alpha(t_{j+1}, t_j) \right] / \hbar \\ & + \sum_{j=1}^{N} \left(\tilde{k}_{\beta j} q_{\beta j} - \tilde{k}_{\alpha j} q_{\alpha j} \right) - (\omega_{\beta j} - \omega_{\alpha j}) \, t_j + \left(\varphi_{\beta j} - \varphi_{\alpha j} \right) \\ & + \left[\tilde{p}_{\beta,D} (q - q_{\beta,D}) - \tilde{p}_{\alpha,D} (q - q_{\alpha,D}) \right] / \hbar \end{aligned} \tag{97}$$

where $S_{\alpha j} = S_\alpha(t_{j+1}, t_j)$ and $S_{\beta j} = S_\beta(t_{j+1}, t_j)$ are the action integrals along α (β) paths; $\hbar k_{\alpha j}(\hbar k_{\beta j})$ are the momenta transferred to the atoms by the j-th beam splitter along the α (β) arm; $q_{\alpha j}$ and $q_{\beta j}$ are the classical coordinates of the centers of the beam splitter/atom interactions; $\omega_{\alpha j}(\omega_{\beta j})$ are the angular frequencies of the e.m. waves; $\varphi_{\alpha j}(\varphi_{\beta j})$ are the fixed phases of the j-th beam splitters; D is the exit port.

With our new approach in (4+1)D the action terms are replaced by the phase jumps induced by the beam splitters along the fourth space coordinate $c\tau$:

$$\sum_{j=1}^{N} c^2 \left[\delta M_{\beta j}\tau_{\beta j} - \delta M_{\alpha j}\tau_{\alpha j}\right]/\hbar \tag{98}$$

in which $\delta M_{\beta j}$ ($\delta M_{\alpha j}$) are the mass changes introduced by each splitter.

A consequence of the existence of a Lagrange invariant along homologous segments of the two branches is that:

(99)
$$\left(\tilde{p}_{\alpha j+1} + \tilde{p}_{\beta j+1}\right)\left(q_{\beta j+1} - q_{\alpha j+1}\right) - \left(\tilde{p}_{\alpha j} + \tilde{p}_{\beta j}\right)\left(q_{\beta j} - q_{\alpha j}\right) - \hbar\left(\tilde{k}_{\beta j} + \tilde{k}_{\alpha j}\right)\left(q_{\beta j} - q_{\alpha j}\right) = 0$$

where the momenta $\tilde{p} = (p_x, p_y, p_z, Mc)$ refer to their values immediately before each beam splitter after which they are increased by $\hbar\tilde{k} = (\hbar k_x, \hbar k_x, \hbar k_x, c\delta M)$.

We get:

$$\begin{aligned}\delta\phi(q) = &\sum_{j=1}^{N}\left(\tilde{k}_{\beta j}q_{\beta j} - \tilde{k}_{\alpha j}q_{\alpha j}\right) - \left(\tilde{k}_{\beta j} + \tilde{k}_{\alpha j}\right)\left(q_{\beta j} - q_{\alpha j}\right)/2\\ &-\sum_{j=1}^{N}(\omega_{\beta j} - \omega_{\alpha j})t_j + \sum_{j=1}^{N}\left(\varphi_{\beta j} - \varphi_{\alpha j}\right)\\ &+\frac{\left(\tilde{p}_{\beta D} - \tilde{p}_{\alpha D}\right)}{\hbar}\left(q - \frac{q_{\beta D} + q_{\alpha D}}{2}\right) - \frac{\tilde{p}_{\alpha 1} + \tilde{p}_{\beta 1}}{2\hbar}\left(q_{\beta 1} - q_{\alpha 1}\right)\end{aligned} \tag{100}$$

Usually one has the same input point for both arms $q_{\beta 1} = q_{\alpha 1}$ and we may use the mid-point theorem [1] which states that the phase difference for the fringe signal integrated over space at the output is given by the phase difference before integration at the mid-point $\left(q_{\beta,D} + q_{\alpha,D}\right)/2$. So that the last line of the previous equation drops out and the phase shift difference $\delta\phi$ between the two arms (α,β) for an interferometer with N beam splitters can be written as:

$$\delta\phi = \sum_{j=1}^{N}\left[\left(\tilde{k}_{\beta j} - \tilde{k}_{\alpha j}\right)q_j - (\omega_{\beta j} - \omega_{\alpha j})t_j + \left(\varphi_{\beta j} - \varphi_{\alpha j}\right)\right] \tag{101}$$

with

$$q_j = \frac{\left(q_{\beta j} + q_{\alpha j}\right)}{2} \tag{102}$$

where the coordinates $q_{\alpha j}$ and $q_{\beta j}$ are calculated with the ABCD matrices. Let us emphasize that this phase-shift formula is manifestly gauge-invariant and that it applies to clocks as well as to systems where internal degrees of freedom are either absent or not used ($\delta M_j = 0$). The same formula gives the resonance condition in an atomic clock and the Sagnac shift in an atomic gyro.

The detailed application to various space-time sensors using atoms or photons: clocks, gravimeters, gradiometers, gyros or gravitational wave detectors will be found in subsequent lecture notes and other publications and follows the lines indicated in references [13, 1, 18]. The WKB method is preferred for continuous systems using fields of constant energy. The ABCD method offers a Fourier transform propagator better suited for pulsed systems. Finally this method can be extended to take Kerr-type interaction terms into account [19].

Appendix A.

Hamiltonian and equations of motion for a massive point particle in General Relativity

In a space-time characterized by the metric tensor $g_{\mu\nu}$(with the signature (+,-,-,-)) the Lagrangian for a point particle having the mass M is :

$$L = -Mc\sqrt{g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}} \tag{A.1}$$

where the dot stands for the time derivative.

The canonical 4-momentum is:

$$p_{\mu} = -\frac{\partial L}{\partial \dot{x}^{\mu}} = Mc\frac{g_{\mu\nu}\dot{x}^{\nu}}{\sqrt{g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}}} = Mcg_{\mu\nu}u^{\nu} \tag{A.2}$$

where u^{ν} is the normalized 4-velocity. The Hamiltonian is:

$$H = \vec{p}\cdot\vec{v} - L = -p_i\dot{x}^i - L = Mc^2\frac{g_{00}c + g_{0i}\dot{x}^i}{\sqrt{g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}}} = p_0 c \tag{A.3}$$

From (A.2) we check that:

$$g^{\mu\nu}p_{\mu}p_{\nu} = M^2c^2 \tag{A.4}$$

which can be solved for p_0 :

$$p_0 = -\frac{g^{0i}p_i}{g^{00}} + \frac{\left(M^2c^2 - f^{ij}p_ip_j\right)^{1/2}}{\sqrt{g^{00}}} \tag{A.5}$$

with([3])

$$f^{ij} = g^{ij} - \frac{g^{0i}g^{0j}}{g^{00}} \tag{A.11}$$

The Euler-Lagrange equations of motion are:

$$\dot{p}_\mu = \frac{1}{2M^*}\partial_\mu g_{\lambda\nu} p^\lambda p^\nu \tag{A.12}$$

to be combined with equation (A.2)

$$\dot{x}^\mu = \frac{1}{M^*} g^{\mu\nu} p_\nu \tag{A.13}$$

expressed with a "relativistic mass":

$$M^* = M\frac{dt}{d\tau} = \frac{Mc}{\sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}} \tag{A.14}$$

where τ is the proper time.

([3]) One can also use (A.2) and (A.3) with:

$$(g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu)^{-1} = \frac{g^{00}}{c^2}\left(1 - \frac{f^{ij}p_ip_j}{M^2c^2}\right) \tag{A.6}$$

which gives:

$$p_0 = g_{i0}p^i + Mcg_{00}\sqrt{g^{00}}\left(1 - \frac{f^{ij}p_ip_j}{M^2c^2}\right)^{1/2} \tag{A.7}$$

To show the equivalence with (A.5) we use:

$$p^i = g^{ij}p_j + g^{i0}p_0 \tag{A.8}$$

$$g^{00}g_{00} = 1 - g_{i0}g^{i0} \tag{A.9}$$

$$g_{i0}g^{ij} = -g_{00}g^{j0} \tag{A.10}$$

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